# From Filtering to Inversion: Kalman-Based Methods for Nonlinear Parameter Estimation in Geotechnics

21st EnKF Workshop 2026

Rosendal, Norway

Jörg Benndorf, **Konstantin Ibadullaev**,
Ching Yin Kwok, Thomas Nagel, Björn Sprungk

17.06.2026

# Content

# Introduction

# Inverse Problem

Consider inverse problem: find parameter $\theta$ given an observation $y$ of the model

$$\boxed{y = \mathcal{G}(\theta) + \eta} \tag{1}$$

where $\mathcal{G} : R^{N_\theta} \to R^{N_y}$ is a forward map from the parameter to the observation space and $\eta \sim \mathcal{N}(0, \Sigma_\eta)$ is the observational noise.

# Optimization approach

From the optimization perspective, the optimal $\theta$ minimizes the loss function:

$$\boxed{\Phi(\theta, y) = \frac{1}{2}\|\Sigma_\eta^{-\frac{1}{2}}(y - \mathcal{G}(\theta))\|^2} \tag{2}$$

When the problem is not well-posed regularization is required:

$$\boxed{\Phi_R(\theta, y) = \Phi(\theta, y) + \frac{1}{2}\|\Sigma_0^{-\frac{1}{2}}(\theta - r_0)\|^2} \tag{3}$$

with $r_0$ and $\Sigma_0$ prior mean and prior covariance matrix, respectively.


Jörg Benndorf, **Konstantin Ibadullaev**, Ching Yin Kwok, Thomas Nagel, Björn Sprungk

Europäische Union
Europa fördert Sachsen.
ESF
Europäischer Sozialfonds
TU BERGAKADEMIE FREIBERG

# Bayesian approach

From the Bayesian point of view, we seek the posterior distribution.

$$\boxed{\rho(\theta|y) = \frac{\rho(y|\theta)\rho_0(\theta)}{\rho(y)} = \frac{1}{Z(y)} e^{-\Phi(\theta,y)}\rho_0(\theta), \;\; with \;\; Z(y) = \int e^{-\Phi(\theta,y)}\rho_0(\theta)d\theta} \tag{4}$$

For a flat prior $\rho_0(\theta)$

$$\boxed{\rho(\theta|y) = \frac{1}{Z(y)} e^{-\Phi(\theta,y)} \propto e^{-\Phi(\theta,y)}}$$

For a Gaussian prior $\rho_0(\theta)$

$$\boxed{\rho(\theta|y) = \frac{1}{Z(y)} e^{-\Phi(\theta,y)}\rho_0(\theta) \propto e^{-\Phi_R(\theta,y)}}$$

$$\boxed{\hat{\theta}_{MAP} = \arg\max_{\theta} \rho(\theta|y) = \arg\min_{\theta} \left[\Phi(\theta,y) - \log \rho_0(\theta)\right]}$$


Jörg Benndorf, **Konstantin Ibadullaev**, Ching Yin Kwok, Thomas Nagel, Björn Sprungk

Europäische Union
Europa fördert Sachsen.
ESF
Europäischer Sozialfonds
TU BERGAKADEMIE FREIBERG

# Coupling Filters and Inversion

# Kalman Filter

Consider the evolution of the state $x_k$ and an observation $y_k$ at time $k$:

$$\boxed{\begin{aligned} x_{k+1} &= f(x_k) + \omega_{k+1}, \quad \omega_{k+1} \sim \mathcal{N}(0, \Sigma_\omega) \\ y_{k+1} &= h(x_{k+1}) + \nu_{k+1}, \quad \nu_{k+1} \sim \mathcal{N}(0, \Sigma_\nu) \end{aligned}} \tag{5}$$

Kalman Filter follows the probabilistic approach and can be viewed as a 2-step procedure leading to the recursive relation $\rho(x_k|Y_k) \rightarrow \rho(x_{k+1}|Y_{k+1})$[1,2].

# Kalman Filter

1. **Prediction step**
   Estimate $\hat{m}_{k+1}, \hat{C}_{k+1}$ of $\rho(x_{k+1}|Y_k)$ given $\rho(x_k|Y_k) \sim \mathcal{N}(m_k, C_k)$

$$\boxed{\begin{aligned} \hat{m}_{k+1} &= \int f(x_k)\rho(x_k|Y_k)dx_k = \mathbb{E}[f(x_k)|Y_k] \\ \hat{C}_{k+1} &= \int [f(x_k)f(x_k)^T + \Sigma_\nu]\rho(x_k|Y_k)dx_k - \hat{m}_{k+1}\hat{m}_{k+1}^T = Cov[f(x_k)|Y_k] + \Sigma_\nu \end{aligned}} \quad (6)$$

2. **Analysis step**

# Kalman Filter

1. **Prediction step**
2. **Analysis step**
   Obtain $\rho(x_{k+1}|Y_{k+1})$ via conditioning of the joint distribution $\rho(x_{k+1}, y_{k+1}|Y_k)$ approximated by Gaussian.

$$
\boxed{
\begin{aligned}
\hat{y}_{k+1} &= \mathbb{E}[h(x_{k+1})|Y_k] \\
\hat{C}^{xy}_{k+1} &= Cov[x_{k+1}, h(x_{k+1})|Y_k] \\
\hat{C}^{yy}_{k+1} &= Cov[h(x_{k+1})|Y_k] + \Sigma_\nu \\
m_{k+1} &= \hat{m}_{k+1} + \hat{C}^{xy}_{k+1}(\hat{C}^{yy}_{k+1})^{-1}(y_{k+1} - \hat{y}_{k+1}) \\
C_{k+1} &= \hat{C}_{k+1} - \hat{C}^{xy}_{k+1}(\hat{C}^{yy}_{k+1})^{-1}(\hat{C}^{xy}_{k+1})^T
\end{aligned}
}
\tag{7}
$$


Jörg Benndorf, **Konstantin Ibadullaev**, Ching Yin Kwok, Thomas Nagel, Björn Sprungk

# Nonlinear Kalman Filters

Different nonlinear Kalman filters are designed to approximate integrals in (7) under nonlinear transformations.

- **Extended Kalman Filter(ExKF)**
  Makes use of Taylor approximations $\rightarrow$ requires explicit gradient computations.
- **Unscented Kalman Filter(UKF)**
  Utilizes a deterministic quadrature rule $\rightarrow$ works with a deterministic set of transformed particles (sigma points).
- **Ensemble Kalman Filter(EKF)**
  Utilizes Monte Carlo sampling $\rightarrow$ computes empirical mean and covariances over an ensemble of particles.


Europäische Union
Europa fördert Sachsen.
ESF
Europäischer Sozialfonds
TU Bergakademie Freiberg

# Coupling Filters and Inversion

If we couple an inverse problem (1) with a dynamical system for the parameter, we obtain a hidden Markov model with $\eta \sim \mathcal{N}(0, \frac{1}{N}\Sigma_\eta)$:

$$\boxed{\begin{aligned} \theta_{n+1} &= \theta_n \\ y_{n+1} &= \mathcal{G}(\theta_{n+1}) + \eta_{n+1} \end{aligned}} \tag{8}$$

We can apply Kalman filters iteratively $N$ times on (8) to approximate the posterior distribution.

$$\boxed{\begin{aligned} \rho_0(\theta) &= \rho_{prior}(\theta) \\ \rho_n(\theta) &\to \rho_{n+1}(\theta) \propto \rho_n(\theta) e^{-\frac{1}{N}\Phi(\theta, y)} \end{aligned}} \tag{9}$$

The process can be generalized by a meta algorithm, **the Gaussian Approximation Algorithm (GAA)**[1,4]

# Gaussian Approximation Algorithm

Consider the following stochastic dynamical system (mean-field system):

$$\boxed{\begin{aligned} \theta_{n+1} &= \alpha\theta_n + (1-\alpha)r_0 + \omega_{n+1} \\ y_{n+1} &= \mathcal{G}(\theta_{n+1}) + \nu_{n+1} \end{aligned}} \tag{10}$$

where $\omega_{n+1} \sim \mathcal{N}(0, \Sigma_\omega)$ and $\nu_{n+1} \sim \mathcal{N}(0, \Sigma_\nu)$ are artificial evolution error in parameter and observation space, and $\alpha$ is a regularization parameter[1,4].

# Gaussian Approximation Algorithm

## Prediction Step

- **Prediction step**

$$\boxed{\begin{aligned}\hat{m}_{n+1} &= \mathbb{E}[\theta_{n+1}|Y_n] = \alpha\theta_n + (1-\alpha)r_0 \\ \hat{C}_{n+1} &= Cov[\theta_{n+1}|Y_n] = \alpha^2 C_n + \Sigma_w\end{aligned}} \tag{11}$$

$\rho_n(\theta) \to \hat{\rho}_{n+1}(\theta)$ with $\rho_n \sim \mathcal{N}(m_n, C_n)$

- **Analysis step**

# Gaussian Approximation Algorithm

## Analysis Step

- **Prediction step**
- **Analysis step**
  We have the following joint Gaussian distribution $\{\theta_{n+1}, y_{n+1}\}|Y_n$:

$$\boxed{\mathcal{N}\left(\begin{bmatrix}\hat{m}_{n+1}\\ \hat{y}_{n+1}\end{bmatrix}, \begin{bmatrix}\hat{C}_{n+1} & \hat{C}^{\theta y}_{n+1}\\ (\hat{C}^{\theta y}_{n+1})^T & \hat{C}^{yy}_{n+1}\end{bmatrix}\right)} \tag{12}$$

$\hat{\rho}_{n+1}(\theta) \to \rho_{n+1}(\theta)$ for $\theta_{n+1}|Y_n$ with $\rho_{n+1} \sim \mathcal{N}(m_{n+1}, C_{n+1})$

# Gaussian Approximation Algorithm

## Analysis Step

1. Projecting the joint distribution:

$$\boxed{\begin{aligned} \hat{y}_{n+1} &= \mathbb{E}[\mathcal{G}(\theta_{n+1})|Y_n] \\ \hat{C}^{\theta y}_{n+1} &= Cov[\theta_{n+1}, \mathcal{G}(\theta_{n+1})|Y_n] \\ \hat{C}^{yy}_{n+1} &= Cov[\mathcal{G}(\theta_{n+1})|Y_n] + \Sigma_\nu \end{aligned}} \tag{13}$$

2. Conditioning the Gaussian to find $\theta_{n+1}|\{Y_n, y_{n+1}\} = \theta_{n+1}|Y_{n+1}$

$$\boxed{\begin{aligned} m_{n+1} &= \hat{m}_{n+1} + \hat{C}^{\theta y}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y_{n+1} - \hat{y}_{n+1}) \\ C_{n+1} &= \hat{C}_{n+1} - \hat{C}^{\theta y}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}\hat{C}^{\theta y T}_{n+1} \end{aligned}} \tag{14}$$

In case of nonlinear $\mathcal{G}$, integrals appear in (13).


Jörg Benndorf, **Konstantin Ibadullaev**, Ching Yin Kwok, Thomas Nagel, Björn Sprungk

Europäische Union
Europa fördert Sachsen. ESF Europäischer Sozialfonds

# Unscented Kalman Inversion

If we apply the Unscented Kalman Filter on the system (10) the GAA looks as follows [1]:

1. **Prediction step**

$$\boxed{\begin{aligned}\hat{m}_{n+1} &= \alpha m_n + (1-\alpha) r_0 \\ \hat{C}_{n+1} &= \alpha^2 C_n + \Sigma_\omega\end{aligned}} \tag{15}$$

2. **Generate sigma points**
3. **Analysis step**

# Unscented Kalman Inversion

The UKI ensemble is deterministic and has the size $J = 2N_\theta + 1$.

1. **Prediction step**
2. **Generate sigma points**

$$\boxed{\begin{aligned}\hat{\theta}^0_{n+1} &= \hat{m}_{n+1}\\ \hat{\theta}^j_{n+1} &= \hat{m}_{n+1} + c[\sqrt{\hat{C}_{n+1}}]_j \ \ (1 \le j \le N_\theta)\\ \hat{\theta}^{j+N_\theta}_{n+1} &= \hat{m}_{n+1} - c[\sqrt{\hat{C}_{n+1}}]_j \ \ (1 \le j \le N_\theta)\end{aligned}} \tag{16}$$

With $a = \min(\sqrt{\frac{4}{N_\theta}}, 1)$, weights $c = a\sqrt{N_\theta}$, $[\sqrt{\hat{C}_{n+1}}]_j$ is column $j$ of Cholesky Factor of $\hat{C}_{n+1}$.

3. **Analysis step**

The integrals appearing in the analysis step for a nonlinear $\mathcal{G}$ are approximated by the deterministic quadrature rule.


Jörg Benndorf, **Konstantin Ibadullaev**, Ching Yin Kwok, Thomas Nagel, Björn Sprungk

Europäische Union
Europa fördert Sachsen. ESF Europäischer Sozialfonds

# Unscented Kalman Inversion

1. **Prediction step**
2. **Generate sigma points**
3. **Prediction step**

$$
\boxed{
\begin{aligned}
&\hat{y}^j_{n+1} = \mathcal{G}(\hat{\theta}^j_{n+1}) \;\; ; \;\; \hat{y}_{n+1} = \hat{y}^0_{n+1} \\
&\hat{C}^{\theta y}_{n+1} = \sum_{j=1}^{2N_\theta} \frac{1}{2a^2 N_\theta} (\hat{\theta}^j_{n+1} - \hat{m}_{n+1})(\hat{y}^j_{n+1} - \hat{y}_{n+1})^T \\
&\hat{C}^{yy}_{n+1} = \sum_{j=1}^{2N_\theta} \frac{1}{2a^2 N_\theta} (\hat{y}^j_{n+1} - \hat{y}_{n+1})(\hat{y}^j_{n+1} - \hat{y}_{n+1})^T + \Sigma_\nu \\
&m_{n+1} = \hat{m}_{n+1} + \hat{C}^{\theta y}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y - \hat{y}_{n+1}) \\
&C_{n+1} = \hat{C}_{n+1} - \hat{C}^{\theta y}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(\hat{C}^{\theta y}_{n+1})^T
\end{aligned}
}
\tag{17}
$$

# Ensemble Kalman Inversion

If we apply the Ensemble Kalman Filter on the system (10) the GAA looks as follows[1,2]:

1. **Prediction step**

$$\boxed{\begin{aligned} \hat{\theta}^j_{n+1} &= \alpha\theta^j_n + (1-\alpha)r_0 + \omega^j_{n+1} \\ \hat{m}_{n+1} &= \frac{1}{J}\sum_{j=1}^{J}\hat{\theta}^j_{n+1} \end{aligned}} \tag{18}$$

2. **Analysis step**

EKI works with a set of particles whose dynamics is predicted by (10) and then used to predict empirical approximations of covariances.
For large $J$ the updates in the analysis step match to those of the GAA.

# Ensemble Kalman Inversion

1. **Prediction step**
2. **Analysis step**

$$
\begin{aligned}
\hat{y}^{j}_{n+1} &= \mathcal{G}(\theta^{j}_{n+1}) \;\; ; \;\; \hat{y}_{n+1} = \frac{1}{J}\sum_{j=1}^{J}\mathcal{G}(\theta^{j}_{n+1}) \\
\hat{C}^{\theta y}_{n+1} &= \frac{1}{J-1}\sum_{j=1}^{J}(\theta^{j}_{n+1} - \hat{m}_{n+1})(\hat{y}^{j}_{n+1} - \hat{y}_{n+1})^{T} \\
\hat{C}^{yy}_{n+1} &= \frac{1}{J-1}\sum_{j=1}^{J}(\hat{y}^{j}_{n+1} - \hat{y}_{n+1})(\hat{y}^{j}_{n+1} - \hat{y}_{n+1})^{T} + \Sigma_{\nu} \\
\theta^{j}_{n+1} &= \hat{\theta}^{j}_{n+1} + \hat{C}^{\theta y}_{n+1}(\hat{C}^{yy}_{n+1})^{-1}(y - \hat{y}^{j}_{n+1} - \nu^{j}{}_{n+1})
\end{aligned}
\tag{19}
$$



Jörg Benndorf, **Konstantin Ibadullaev**, Ching Yin Kwok, Thomas Nagel, Björn Sprungk

# Comparison

| Feature | EKI | UKI |
|---|---|---|
| Ensemble type | Stochastic | Deterministic |
| Ensemble collapse | Yes | No |
| Low-rank approximation | Yes | No |

When to use

- **EKI**: High-dimensional problems, covariance estimate is of a lesser interest.
- **UKI**: Low- to moderate-dimensional problems, we assume that the Gaussian-like uncertainty is plausible.

The choice of the inversion method is a tradeoff.

# Inverse Problems in Geotechnical Engineering

# Stiffness Inversion

The forward model $\mathcal{G}$ computes the surface uplift $u$(scaled for cm) for a vertical slice of a homogeneous rock layer split into $n$ cells given stiffness $E$, porosity $\mu$, bulk weight $\gamma_w$, discretization step $\Delta z$.

$$\boxed{Y = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_i \\ \vdots \\ u_n \end{pmatrix} = (1-\mu)\gamma_w \Delta z^2 \begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 1 & \frac{1}{2} & 0 & \cdots & 0 \\ 1 & 1 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{E_1} \\ \frac{1}{E_2} \\ \vdots \\ \frac{1}{E_n} \end{pmatrix}} \tag{20}$$

We have noisy uplift observations with the noise $\eta \sim \mathcal{N}(0, \gamma I_{N_y}),\ \gamma = 1\ mm^2$, and assume $\theta = \ logE$ is normally distributed.
Hence, $\mathcal{G}(\phi(\theta))$, with a nonlinear transformation $\phi(\theta) = \exp(\theta)$.

# Stiffness Inversion

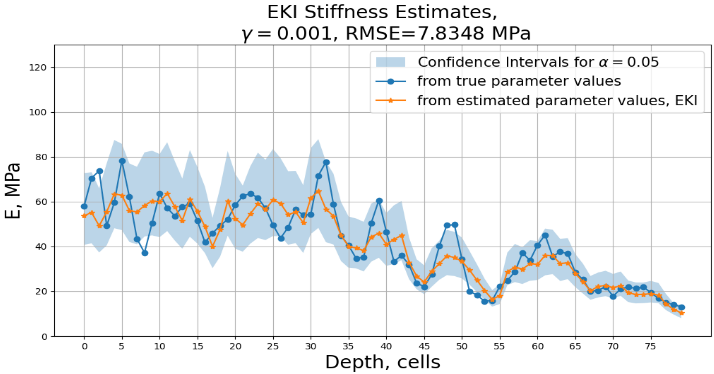


Figure 1: Estimated stiffness $E$.

# Stiffness Inversion

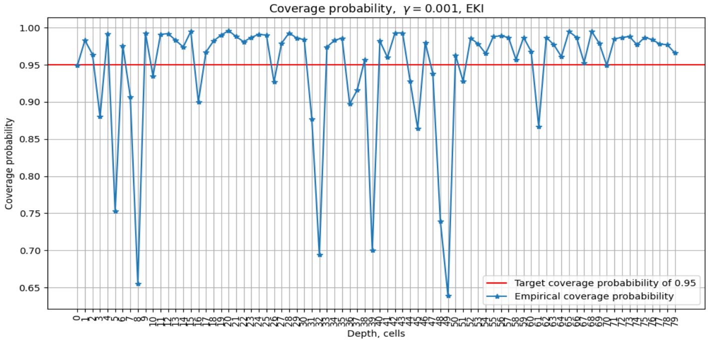


Figure 2: Coverage probability.

# Inversion of Gas Cavern Parameters

We use the following model for modelling a subsidence induced by multiple caverns of a cylindrical shape[3].

$$
\boxed{\begin{aligned}
&s_{total}(r,t) = \sum_{i=0}^{N} s_i(r,t) = \sum_{i=0}^{N} s_i^{max}(t) exp[-\pi \frac{r(x,y)_i^2}{R_i^2}] \\
&R_i = \frac{\sqrt{H_i^f \cdot H_i^r}}{tan\beta_i} \; ; \; s_i^{max}(t) = \frac{\Delta k(t)_i V_i^0}{R_i^2}
\end{aligned}} \tag{21}
$$

The goal is to estimate individual influence angles $\beta_i$ and current convergence ratio $k(t)$ for each period given noisy yearly subsidence observations $s$ with the noise $\eta \sim \mathcal{N}(0, \gamma I_{N_y}),\ \gamma = 1\ cm^2$, initial cavern volumes $V_i^0$, geometric parameters.

# Inversion of Gas Cavern Parameters

The inverse problem is solved for 3 caverns having 63 parameters in total: 3 influence angles $\beta_i$ and 60 convergence volumes for observation period $t$ of 20 years.

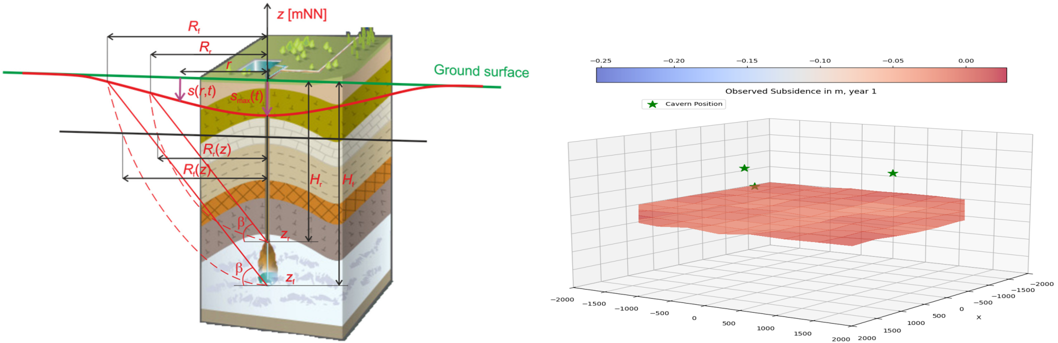


Figure 3: Gas caverns and observed subsidence.

# Inversion of Gas Cavern Parameters

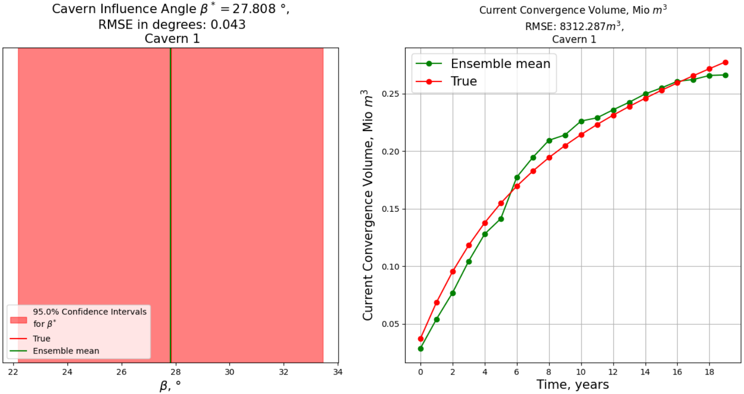


Figure 4: EKI estimates for $\beta$s and $k$s

# Inversion of Gas Cavern Parameters

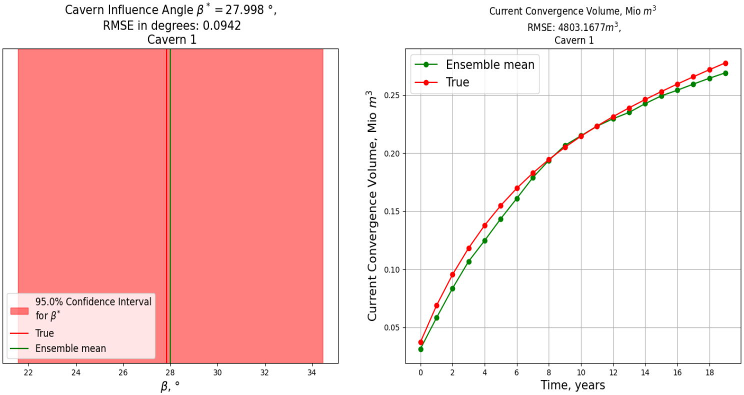


Figure 5: UKI estimates for $\beta$s and $k$s.

# Inversion of Gas Cavern Parameters

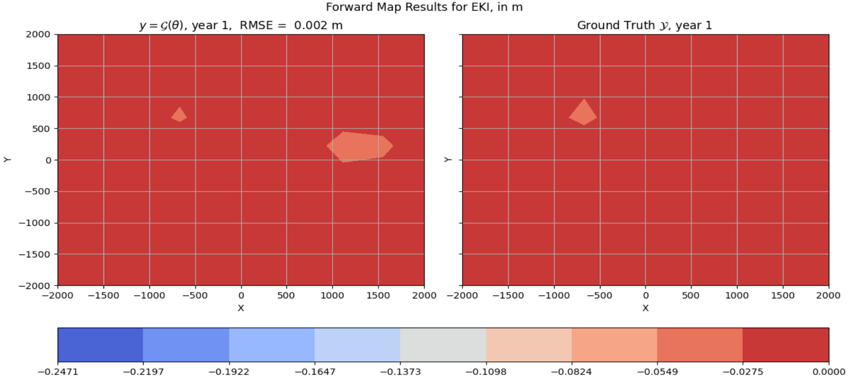


Figure 6: $\mathcal{G}(\phi(\beta^*, k^*))_{EKI}$ compared to noise-free subsidence.

# Inversion of Gas Cavern Parameters

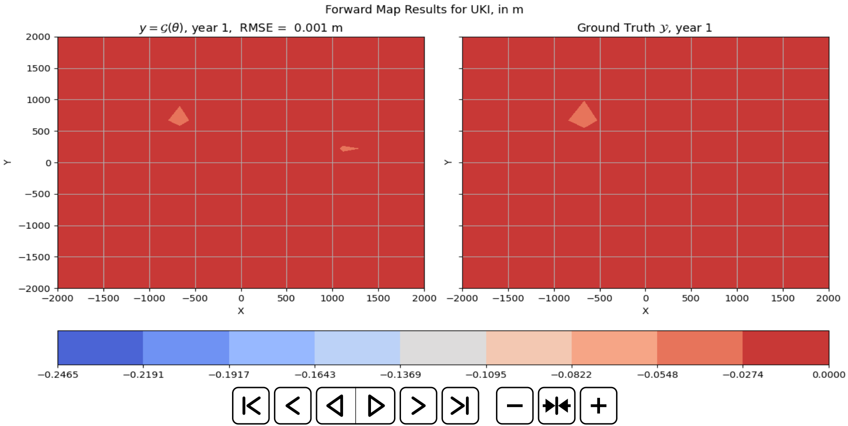


Figure 7: $\mathcal{G}(\phi(\beta^*, k^*))_{UKI}$ compared to noise-free subsidence.

# Summary

- Both UKI and EKI provide good parameter estimates for the presented nonlinear inverse problems.
- Both UKI and EKI provide gaussian-like uncertainty, which is valid under particular assumptions for parameter distributions and inverse problem.
- The quality of the obtained confidence intervals is questionable.
  EKI tends to underestimate the uncertainty due to ensemble collapse, whereas UKI yields a conservative estimate (too wide intervals).
- The choice of the inversion method depends on the dimensionality of the parameter space and assumptions about the parameter distribution.

# Thank you for your attention!

# References


1. D. Z. Huang, T. Schneider, and A. M. Stuart,2022. Iterated Kalman Methodology For Inverse Problems / Unscented Kalman Inversion, Journal of Computational Physics 463 (2022) 111262
2. M. A. Iglesias, K. J H Law and A. M. Stuart, 2013. Ensemble Kalman methods for inverse problems, Inverse Problems. 29 (4) (2013) 045001.
3. R. Misa, A. Sroka et al, 2023. Determination of convergence of underground gas storage caverns using non-invasive methodology based on land surface subsidence measurement, Journal of Rock Mechanics and Geotechnical Engineering V. 15, 8, 1944-1950.
4. InverseProblems.jl https://github.com/PKU-CMEGroup/InverseProblems.jl/tree/master

# Supplementary slides

# Supplementary slides

## Modified Unscented Transform[1,4]

Let $\mathcal{G}_i, i = 1, 2$ denote any pair of real vector-valued functions on $\mathbb{R}^{N_\theta}$, $\theta^0 = m$
If $\theta \sim \mathcal{N}(m, C)$ then

$$\mathbb{E}\mathcal{G}_i(\theta) = \mathcal{G}_i(\theta^0) + \mathcal{O}(\|C\|)$$
$$Cov(\mathcal{G}_1(\theta), \mathcal{G}_1(\theta)) = \sum_{j=1}^{2N_\theta} W_j^c(\mathcal{G}_1(\theta^j) - \mathbb{E}\mathcal{G}_1(\theta^j))(\mathcal{G}_2(\theta^j) - \mathbb{E}\mathcal{G}_2(\theta^j))^T + \mathcal{O}(\|C\|^2) \tag{22}$$

thus the modified unscented transform is first and second order accurate in approximating means and covariances of $\mathcal{G}_1(\theta)$ and $\mathcal{G}_2(\theta)$ with respect to small $(\|C\|)$. Furthermore, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are linear then the modified unscented transform is exact for these quantities.

# Supplementary slides

## Stiffness Inversion

- The layer is split into $d = 80$ cells with stiffness $E_i$, bulk weight $\gamma_w$, porosity $\mu$. We want to solve an inverse problem for stiffness $E_i$ given noisy observations for uplift $y_i = u_i + \eta$, where $\eta \sim \mathcal{N}(0, \gamma I_d)$,with $\gamma = 1$ mm.
- We assume that $\log(E)$ is normally distributed with $m_0 = \ln(range(10MPa, ..80MPa))_d$ and $C_0 = \sigma^2 exp(-2\frac{\|d_i - d_j\|}{\lambda})_{d\times d}$ with $\sigma = 0.2$ and $\lambda = 1.8$.
- For EKI we initialize an ensemble of particles $J = 800$ drawn from $\mathcal{N}(m_0, C_0)$.
- For UKI we set prior mean $m_0$ and prior covariance $C_0$, $J = 161$.
- The forward map is updated as follows $\mathcal{G}(\phi(\theta)) = \mathcal{G}(exp(\theta))$.


Europäische Union
Europa fördert Sachsen.
ESF
Europäischer Sozialfonds
TU BERGAKADEMIE FREIBERG

# Supplementary slides

## Stiffness Inversion

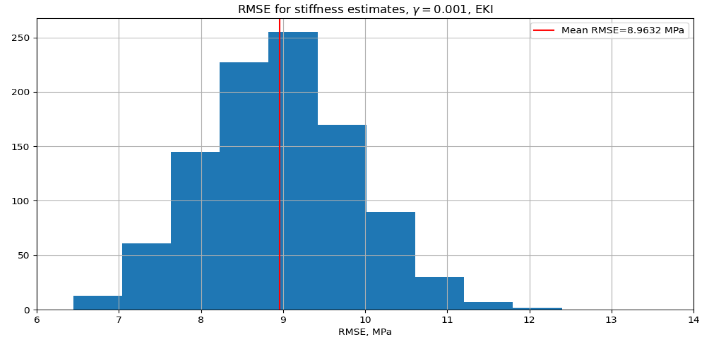


Figure 8: RMSE for $E^*$.

# Supplementary slides

## Stiffness Inversion

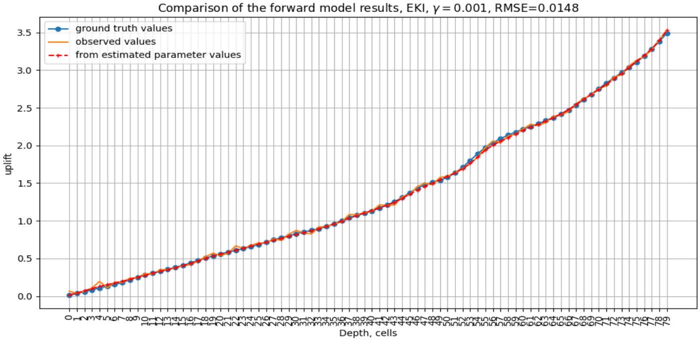


Figure 9: Observed Uplift vs $\mathcal{G}(exp(logE^*))$.

- We assume observational noise $\eta \sim \mathcal{N}(0, \gamma I_{gs^2 \times t})$ with $\gamma = 1$cm, $gs = 10$, $t = 20$ years.
- The prior for $\beta_i$s assumed to be multivariate $U(a, b)$ with $a = 25°$, and $b = 50°$, and for converged volumes $k_i$s we set multivariate $\mathcal{N}(0.2_{60}, 0.1 I_{60})$.
- The prior is $U(25, 50) \otimes \mathcal{N}(0.2_{60}, 0.1 I_{60})$.
- The ensemble sizes are $J = 189$ and $J = 127$ for EKI and UKI, respectively.

# Supplementary slides

## Inversion of Gas Cavern Parameters

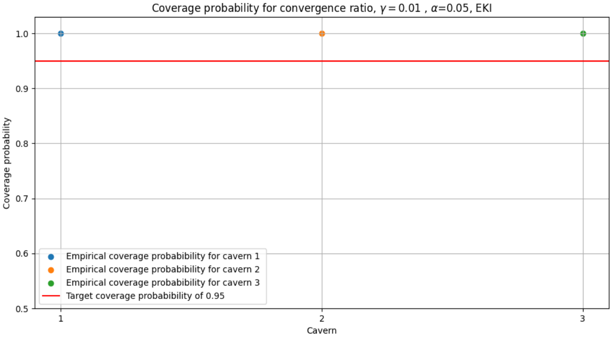


Figure 10: Coverage probability for $\beta$.

# Supplementary slides

## Inversion of Gas Cavern Parameters

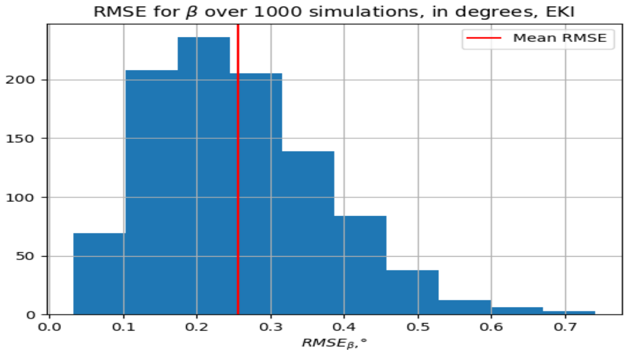


Figure 11: RMSE for $\beta$ and $k$ for EKI.

# Supplementary slides

## Inversion of Gas Cavern Parameters

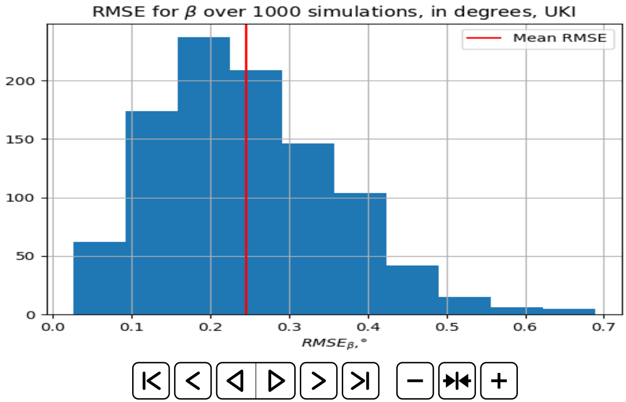


Figure 12: RMSE for $\beta$ and $k$ for UKI.