Dynamical Low-Rank Approximations for Kalman filtering

Thomas Trigo Trindade

21st International EnKF Workshop — June 15th, 2026

جامعة الملك عبدالله للعلوم والتقنية
King Abdullah University of Science and Technology

Collaborators

Fabio Nobile (EPFL)

Sébastien Riffaud (EPFL)

Table of Contents

Continuous Data Assimilation in a nutshell

Dynamics + observations

$$\mathrm{d}\mathcal{X}_t^{\mathrm{signal}} = (A\mathcal{X}_t^{\mathrm{signal}} + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\widetilde{\mathcal{W}}_t \qquad \text{(state)}$$

$$\mathrm{d}\mathcal{Z}_t = H\mathcal{X}_t^{\mathrm{signal}}\mathrm{d}t + \Gamma^{1/2}\mathrm{d}\widetilde{\mathcal{V}}_t \qquad \text{(observations)}$$

$$\mathcal{X}_0^{\mathrm{signal}} \sim \mathcal{N}(m_0, P_0) \qquad \text{(Gaussian initial condition)}$$

where $A \in \mathrm{Mat}(d \times d)$, $b \in \mathbb{R}^d$ and $\Sigma \in \mathrm{Mat}_0(d)$, $H \in \mathrm{Mat}(k \times d)$ and $\Gamma \in \mathrm{Mat}_+(k)$

Filtering problem

Aim : characterise

$$\eta_t = \mathrm{Law}(\mathcal{X}_t^{\mathrm{signal}} \,|\, \mathcal{A}_{\mathcal{Z}_t}) \quad \mathcal{A}_{\mathcal{Z}_t} = \sigma(\mathcal{Z}_s, 0 \leq s \leq t)$$

- $\mathcal{X}_t^{\mathrm{signal}} \in \mathbb{R}^d$, $\mathcal{Z}_t \in \mathbb{R}^k$
- state space $d \gg k$ observation dimension

Continuous Data Assimilation in a nutshell

Dynamics + observations

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\mathrm{signal}} &= (A\boldsymbol{\mathcal{X}}_t^{\mathrm{signal}} + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_t && \text{(state)}\\
\mathrm{d}\boldsymbol{\mathcal{Z}}_t &= H\boldsymbol{\mathcal{X}}_t^{\mathrm{signal}}\mathrm{d}t + \Gamma^{1/2}\mathrm{d}\widetilde{\boldsymbol{\mathcal{V}}}_t && \text{(observations)}\\
\boldsymbol{\mathcal{X}}_0^{\mathrm{signal}} &\sim \boldsymbol{\mathcal{N}}(m_0, P_0) && \text{(Gaussian initial condition)}
\end{aligned}
$$

where $A \in \mathrm{Mat}(d \times d)$, $b \in \mathbb{R}^d$ and $\Sigma \in \mathrm{Mat}_0(d)$, $H \in \mathrm{Mat}(k \times d)$ and $\Gamma \in \mathrm{Mat}_+(k)$

Filtering problem

Aim : characterise

$$\boldsymbol{\eta}_t = \mathrm{Law}(\boldsymbol{\mathcal{X}}_t^{\mathrm{signal}} \mid \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t}) \quad \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t} = \sigma(\boldsymbol{\mathcal{Z}}_s,\, 0 \leq s \leq t)$$

- $\boldsymbol{\mathcal{X}}_t^{\mathrm{signal}} \in \mathbb{R}^d$, $\boldsymbol{\mathcal{Z}}_t \in \mathbb{R}^k$
- state space $d \gg k$ observation dimension

Kalman Filtering

- assume affine dynamics + linear observations + Gaussian initial condition

Theorem (Bucy, Kalman)

Filtering density $\boldsymbol{\eta}_t$ is **Gaussian**, with mean $m_t \in \mathbb{R}^d$ and covariance $P_t \in \mathrm{Mat}(d \times d)$ verifying

$$\mathrm{d}m_t = (Am_t + f)\mathrm{d}t + P_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - Hm_t\mathrm{d}t) \qquad \text{(Mean equation)}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t = AP_t + P_t A - P_t S P_t + \Sigma \qquad \text{(Riccati equation)}$$

- Approach 1 : simulate (m_t, P_t) directly
- Approach 2 : consider *auxiliary system* with $\boldsymbol{\mathcal{X}}_t$ with $\boldsymbol{\mu}_t = \mathrm{Law}(\boldsymbol{\mathcal{X}}_t \,|\, \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t}) = \boldsymbol{\eta}_t$

Example:

(Vanilla) Kalman-Bucy Process (KBP)

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t = (A\boldsymbol{\mathcal{X}}_t + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\boldsymbol{\mathcal{W}}_t + P_t H^\top \Gamma^{-1}\left[\mathrm{d}\boldsymbol{\mathcal{Z}}_t - (H\boldsymbol{\mathcal{X}}_t\mathrm{d}t + \Gamma^{1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t)\right]$$

- framework (almost) adapted for simulation-based filtering

Kalman Filtering

- assume affine dynamics + linear observations + Gaussian initial condition

Theorem (Bucy, Kalman)

Filtering density $\boldsymbol{\eta}_t$ is **Gaussian**, with mean $m_t \in \mathbb{R}^d$ and covariance $P_t \in \mathrm{Mat}(d \times d)$ verifying

$$\mathrm{d}m_t = (Am_t + f)\mathrm{d}t + P_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - Hm_t\mathrm{d}t) \qquad \text{(Mean equation)}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t = AP_t + P_tA - P_tSP_t + \Sigma \qquad \text{(Riccati equation)}$$

- Approach 1 : simulate (m_t, P_t) directly
- Approach 2 : consider *auxiliary system* with $\boldsymbol{\mathcal{X}}_t$ with $\boldsymbol{\mu}_t = \mathrm{Law}(\boldsymbol{\mathcal{X}}_t \,|\, \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t}) = \boldsymbol{\eta}_t$

Example:

(Vanilla) Kalman-Bucy Process (KBP)

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t = (A\boldsymbol{\mathcal{X}}_t + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\boldsymbol{\mathcal{W}}_t + P_tH^\top\Gamma^{-1}\left[\mathrm{d}\boldsymbol{\mathcal{Z}}_t - (H\boldsymbol{\mathcal{X}}_t\mathrm{d}t + \Gamma^{1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t)\right]$$

- framework (almost) adapted for simulation-based filtering

Ensemble Kalman Filtering

Let $\{\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)}\}_{p=1}^N$ an ensemble of N particles, $\{\widehat{\boldsymbol{\mathcal{W}}}_t, \widehat{\boldsymbol{\mathcal{V}}}_t\}_{p=1}^N$ independent BMs; denote

- $\mathbb{E}_N[\widehat{\boldsymbol{\mathcal{X}}}_t] = \frac{1}{N}\sum_{i=1}^N \widehat{\boldsymbol{\mathcal{X}}}_t^{(i)}$ (sample mean)
- $\widehat{P}_t = \frac{N}{N-1}\mathbb{E}_N[(\widehat{\boldsymbol{\mathcal{X}}}_t - \mathbb{E}_N[\widehat{\boldsymbol{\mathcal{X}}}_t])(\widehat{\boldsymbol{\mathcal{X}}}_t - \mathbb{E}_N[\widehat{\boldsymbol{\mathcal{X}}}_t])^\top]$ (sample covariance)

(Vanilla) Ensemble-Kalman Filter

$$\begin{aligned} \mathrm{d}\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} &= (A\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{W}}}_t^{(p)} + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)}\mathrm{d}t - \Gamma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{V}}}_t^{(p)}) \\ \widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} &\sim \boldsymbol{\mathcal{N}}(m_0, P_0), \qquad p = 1, \ldots, N \end{aligned}$$

- some shortcomings
  1. often numerically unstable (combined w/ other techniques to stabilise)
  2. cost of simulating N particles $\propto \mathcal{O}(dN)$ $\rightsquigarrow$ low-rank structure NOT leveraged
- BUT : very good starting point for high-dimensional filtering problems
  1. practical
  2. (often) more robust than particle filters
  3. *satisfactory accuracy even with few particles* $\rightsquigarrow$ low-rank structure !

Indeed : $\mathrm{rank}(\widehat{P}_t) \leq N$, typically $N \ll d$

Ensemble Kalman Filtering

Let $\{\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)}\}_{p=1}^N$ an ensemble of N particles, $\{\widehat{\boldsymbol{\mathcal{W}}}_t, \widehat{\boldsymbol{\mathcal{V}}}_t\}_{p=1}^N$ independent BMs; denote

- $\mathbb{E}_N[\widehat{\boldsymbol{\mathcal{X}}}_t] = \frac{1}{N}\sum_{i=1}^N \widehat{\boldsymbol{\mathcal{X}}}_t^{(i)}$ (sample mean)
- $\widehat{P}_t = \frac{N}{N-1}\mathbb{E}_N[(\widehat{\boldsymbol{\mathcal{X}}}_t - \mathbb{E}_N[\widehat{\boldsymbol{\mathcal{X}}}_t])(\widehat{\boldsymbol{\mathcal{X}}}_t - \mathbb{E}_N[\widehat{\boldsymbol{\mathcal{X}}}_t])^\top]$ (sample covariance)

(Vanilla) Ensemble-Kalman Filter

$$\mathrm{d}\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} = (A\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{W}}}_t^{(p)} + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)}\mathrm{d}t - \Gamma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{V}}}_t^{(p)})$$
$$\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} \sim \boldsymbol{\mathcal{N}}(m_0, P_0), \qquad p = 1, \dots, N$$

- some shortcomings
  1. often numerically unstable (combined w/ other techniques to stabilise)
  2. cost of simulating N particles $\propto \mathcal{O}(dN)$ $\rightsquigarrow$ low-rank structure NOT leveraged
- BUT : very good starting point for high-dimensional filtering problems
  1. practical
  2. (often) more robust than particle filters
  3. *satisfactory accuracy even with few particles* $\rightsquigarrow$ low-rank structure !

  Indeed : $\mathrm{rank}(\widehat{P}_t) \leq N$, typically $N \ll d$

Ensemble Kalman Filtering

Let $\{\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)}\}_{p=1}^N$ an ensemble of N particles, $\{\widehat{\boldsymbol{\mathcal{W}}}_t, \widehat{\boldsymbol{\mathcal{V}}}_t\}_{p=1}^N$ independent BMs; denote

- $\mathbb{E}_N[\widehat{\boldsymbol{\mathcal{X}}}_t] = \frac{1}{N}\sum_{i=1}^N \widehat{\boldsymbol{\mathcal{X}}}_t^{(i)}$ (sample mean)
- $\widehat{P}_t = \frac{N}{N-1}\mathbb{E}_N[(\widehat{\boldsymbol{\mathcal{X}}}_t - \mathbb{E}_N[\widehat{\boldsymbol{\mathcal{X}}}_t])(\widehat{\boldsymbol{\mathcal{X}}}_t - \mathbb{E}_N[\widehat{\boldsymbol{\mathcal{X}}}_t])^\top]$ (sample covariance)

(Vanilla) Ensemble-Kalman Filter

$$
\begin{aligned}
\mathrm{d}\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} &= (A\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{W}}}_t^{(p)} + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)}\mathrm{d}t - \Gamma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{V}}}_t^{(p)}) \\
\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} &\sim \boldsymbol{\mathcal{N}}(m_0, P_0), \qquad p = 1, \ldots, N
\end{aligned}
$$

- some shortcomings
  1. often numerically unstable (combined w/ other techniques to stabilise)
  2. cost of simulating N particles $\propto \mathcal{O}(dN)$ $\rightsquigarrow$ low-rank structure NOT leveraged
- BUT : very good starting point for high-dimensional filtering problems
  1. practical
  2. (often) more robust than particle filters
  3. *satisfactory accuracy even with few particles* $\rightsquigarrow$ low-rank structure !

     Indeed : $\mathrm{rank}(\widehat{P}_t) \leq N$, typically $N \ll d$

Dynamical Low-Rank Approximations

- $\boldsymbol{\mathcal{X}}_t^{\text{true}} \in L^2(\Omega, \mathbb{R}^d) \implies \boldsymbol{\mathcal{X}}_t^{\text{true}} = \mathbb{E}[\boldsymbol{\mathcal{X}}_t^{\text{true}}] + \sum_{i=1}^{\infty} \sigma_i(t) \boldsymbol{U}_i(t) \tilde{\boldsymbol{Y}}_i(t)$, where $\boldsymbol{U}_i(t) \in \mathbb{R}^d$, $\tilde{\boldsymbol{Y}}_i(t, \omega) \in L^2(\Omega, \mathbb{R})$
- if $\sigma_i(t) \downarrow 0$ quickly, then

$$\boldsymbol{\mathcal{X}}_t^R = \mathbb{E}[\boldsymbol{\mathcal{X}}_t^{\text{true}}] + \sum_{i=1}^{R} \boldsymbol{U}_i(t) \boldsymbol{Y}_i(t) \approx \boldsymbol{\mathcal{X}}_t^{\text{true}},$$

where $\boldsymbol{Y}_i(t) = \sigma_i(t) \tilde{\boldsymbol{Y}}_i(t) \rightsquigarrow$ potentially important computational gains cost $\propto \mathcal{O}(R(d+N))$ vs $\mathcal{O}(dN)$

Low-rank ansatz + evolution equations

Suppose that $\boldsymbol{\mathcal{X}}_t = \sum_{i=1}^{R} \boldsymbol{U}_i(t) \boldsymbol{Y}_i(t)$ is the solution to

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t = a(t, \boldsymbol{\mathcal{X}}_t)\mathrm{d}t + b(t, \boldsymbol{\mathcal{X}}_t)\mathrm{d}\boldsymbol{\mathcal{W}}_t$$

Then, by [KNZ25]

$$\mathrm{d}\boldsymbol{U}_t = (\boldsymbol{I} - \boldsymbol{U}_t \boldsymbol{U}_t^\top) a(t, \boldsymbol{\mathcal{X}}_t)\mathrm{d}t$$
$$\mathrm{d}\boldsymbol{Y}_t = \boldsymbol{U}_t^\top a(t, \boldsymbol{\mathcal{X}}_t)\mathrm{d}t + \boldsymbol{U}_t^\top b(t, \boldsymbol{\mathcal{X}}_t)\mathrm{d}\boldsymbol{\mathcal{W}}_t$$

- instance of *dynamical low-rank approximations* : exist for matricial ODEs, random PDEs, *SDEs*

Dynamical Low-Rank Approximations

- $\boldsymbol{\mathcal{X}}_t^{\text{true}} \in L^2(\Omega, \mathbb{R}^d) \implies \boldsymbol{\mathcal{X}}_t^{\text{true}} = \mathbb{E}[\boldsymbol{\mathcal{X}}_t^{\text{true}}] + \sum_{i=1}^{\infty} \sigma_i(t) \boldsymbol{U}_i(t) \tilde{\boldsymbol{Y}}_i(t)$, where $\boldsymbol{U}_i(t) \in \mathbb{R}^d$, $\tilde{\boldsymbol{Y}}_i(t, \omega) \in L^2(\Omega, \mathbb{R})$
- if $\sigma_i(t) \downarrow 0$ quickly, then

$$\boldsymbol{\mathcal{X}}_t^R = \mathbb{E}[\boldsymbol{\mathcal{X}}_t^{\text{true}}] + \sum_{i=1}^{R} \boldsymbol{U}_i(t) \boldsymbol{Y}_i(t) \approx \boldsymbol{\mathcal{X}}_t^{\text{true}},$$

where $\boldsymbol{Y}_i(t) = \sigma_i(t) \tilde{\boldsymbol{Y}}_i(t) \rightsquigarrow$ potentially important computational gains cost $\propto \mathcal{O}(R(d+N))$ vs $\mathcal{O}(dN)$

Low-rank ansatz + evolution equations

Suppose that $\boldsymbol{\mathcal{X}}_t = \sum_{i=1}^{R} \boldsymbol{U}_i(t) \boldsymbol{Y}_i(t)$ is the solution to

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t = a(t, \boldsymbol{\mathcal{X}}_t)\mathrm{d}t + b(t, \boldsymbol{\mathcal{X}}_t)\mathrm{d}\boldsymbol{\mathcal{W}}_t$$

Then, by [KNZ25]

$$\mathrm{d}\boldsymbol{U}_t = (\boldsymbol{I} - \boldsymbol{U}_t \boldsymbol{U}_t^\top) a(t, \boldsymbol{\mathcal{X}}_t)\mathrm{d}t$$
$$\mathrm{d}\boldsymbol{Y}_t = \boldsymbol{U}_t^\top a(t, \boldsymbol{\mathcal{X}}_t)\mathrm{d}t + \boldsymbol{U}_t^\top b(t, \boldsymbol{\mathcal{X}}_t)\mathrm{d}\boldsymbol{\mathcal{W}}_t$$

- instance of *dynamical low-rank approximations* : exist for matricial ODEs, random PDEs, *SDEs*

Table of Contents

DLR Kalman-Bucy Process (DLR-KBP)

Goal: derive a *principled* DLR-type reduction for Kalman-Bucy Processes $\rightsquigarrow$ extend to EnKF systems

Strategy

- Step 1: *Generalise* DLR-SDE formulation [KNZ25] to processes with additional Itô process, i.e.

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\text{true}} = a(t,\omega)\mathrm{d}t + b(t,\omega)\mathrm{d}\overline{\boldsymbol{\mathcal{W}}}_t + c(t,\omega)\mathrm{d}\boldsymbol{\mathcal{Z}}_t$$

- Step 2: *Apply* DLR-SDE equations to Kalman-Bucy Process (KBP)

DLR-KBP equations

The modes of DLR-KBP $\mathcal{X}_t^{\mathrm{DLR-KBP}} = U_t^0 + \sum_{i=1}^R U_t^i Y_t^i$ are characterised by

$$\mathrm{d}U_t^0 = (AU_t^0 + f)\mathrm{d}t + P_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0 \mathrm{d}t)$$

$$\mathrm{d}\boldsymbol{U}_t = P_{\boldsymbol{U}_t}^\perp A \boldsymbol{U}_t \mathrm{d}t$$

$$\mathrm{d}\boldsymbol{Y}_t^\top = \boldsymbol{U}_t^\top (A - P_t S)\boldsymbol{U}_t \boldsymbol{Y}_t^\top \mathrm{d}t + \boldsymbol{U}_t^\top \Sigma^{1/2}\mathrm{d}\boldsymbol{\mathcal{W}}_t - \boldsymbol{U}_t^\top P_t H^\top \Gamma^{-1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t$$

- $\boldsymbol{\mathcal{Z}}_t$ does not appear in equations for $\boldsymbol{U}_t$ and $\boldsymbol{Y}_t$

DLR Kalman-Bucy Process (DLR-KBP)

Goal: derive a *principled* DLR-type reduction for Kalman-Bucy Processes $\rightsquigarrow$ extend to EnKF systems

Strategy

- Step 1: *Generalise* DLR-SDE formulation [KNZ25] to processes with additional Itô process, i.e.
$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\text{true}} = a(t,\omega)\mathrm{d}t + b(t,\omega)\mathrm{d}\overline{\boldsymbol{\mathcal{W}}}_t + c(t,\omega)\mathrm{d}\boldsymbol{\mathcal{Z}}_t$$
- Step 2: *Apply* DLR-SDE equations to Kalman-Bucy Process (KBP)

DLR-KBP equations

The modes of DLR-KBP $\boldsymbol{\mathcal{X}}_t^{\text{DLR}-\text{KBP}} = U_t^0 + \sum_{i=1}^R U_t^i Y_t^i$ are characterised by

$$\mathrm{d}U_t^0 = (AU_t^0 + f)\mathrm{d}t + P_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0\mathrm{d}t)$$
$$\mathrm{d}\boldsymbol{U}_t = P_{\boldsymbol{U}_t}^\perp A\boldsymbol{U}_t\mathrm{d}t$$
$$\mathrm{d}\boldsymbol{Y}_t^\top = \boldsymbol{U}_t^\top (A - P_t S)\boldsymbol{U}_t \boldsymbol{Y}_t^\top \mathrm{d}t + \boldsymbol{U}_t^\top \Sigma^{1/2}\mathrm{d}\boldsymbol{\mathcal{W}}_t - \boldsymbol{U}_t^\top P_t H^\top \Gamma^{-1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t$$

- $\boldsymbol{\mathcal{Z}}_t$ does not appear in equations for $\boldsymbol{U}_t$ and $\boldsymbol{Y}_t$

Gaussian Characterisation + Reduced Kalman-Bucy Equations

- the DLR-KBP solution is an inhomogeneous Ornstein-Uhlenbeck process $\rightsquigarrow$ **Gaussian**
- An efficient characterisation is possible, since P_t low-rank by construction

$$P_t = \boldsymbol{U}_t M_{\boldsymbol{Y}_t} \boldsymbol{U}_t^\top \text{ where } M_{\boldsymbol{Y}_t} = \mathbb{E}[\boldsymbol{Y}_t \boldsymbol{Y}_t^\top] \in \mathrm{Mat}(R \times R)$$

Reduced Kalman-Bucy equations

$$\mathrm{d}U_t^0 = AU_t^0 \mathrm{d}t + P_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0 \mathrm{d}t) \qquad \text{(mean, } d \times 1)$$

$$\mathrm{d}\boldsymbol{U}_t = (I - \boldsymbol{U}_t \boldsymbol{U}_t^\top) A \boldsymbol{U}_t \mathrm{d}t \qquad \text{(subspace evolution, } d \times R)$$

$$\dot{M}_{\boldsymbol{Y}_t} = A_{\boldsymbol{U}_t} M_{\boldsymbol{Y}_t} + M_{\boldsymbol{Y}_t} A_{\boldsymbol{U}_t}^\top - M_{\boldsymbol{Y}_t} S_{\boldsymbol{U}_t} M_{\boldsymbol{Y}_t} + \Sigma_{\boldsymbol{U}_t} \qquad \text{(reduced Riccati equation, } R \times R)$$

where $S = H^\top \Gamma^{-1} H$, $A_{\boldsymbol{U}_t} = \boldsymbol{U}_t^\top A \boldsymbol{U}_t$, $S_{\boldsymbol{U}_t} = \boldsymbol{U}_t^\top S \boldsymbol{U}_t$, $\Sigma_{\boldsymbol{U}_t} = \boldsymbol{U}_t^\top \Sigma \boldsymbol{U}_t$

- Practical way to solve moment equations in low-rank fashion
- Equations have already been proposed in [YO21] (and subsequent works) + connections to work in [SHNT23]
- (A symmetric) $\lim_{t\to\infty} \boldsymbol{U}_t = R$ dom. eigenmodes of A — connection to Assimilation in Unstable Subspace [BGA+17]

Gaussian Characterisation + Reduced Kalman-Bucy Equations

- the DLR-KBP solution is an inhomogeneous Ornstein-Uhlenbeck process $\rightsquigarrow$ **Gaussian**
- An efficient characterisation is possible, since P_t low-rank by construction

$$P_t = \boldsymbol{U}_t M_{\boldsymbol{Y}_t} \boldsymbol{U}_t^\top \text{ where } M_{\boldsymbol{Y}_t} = \mathbb{E}[\boldsymbol{Y}_t \boldsymbol{Y}_t^\top] \in \mathrm{Mat}(R \times R)$$

Reduced Kalman-Bucy equations

$$\mathrm{d}U_t^0 = AU_t^0 \mathrm{d}t + P_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0 \mathrm{d}t) \qquad \text{(mean, } d \times 1\text{)}$$

$$\mathrm{d}\boldsymbol{U}_t = (I - \boldsymbol{U}_t \boldsymbol{U}_t^\top) A \boldsymbol{U}_t \mathrm{d}t \qquad \text{(subspace evolution, } d \times R\text{)}$$

$$\dot{M}_{\boldsymbol{Y}_t} = A_{\boldsymbol{U}_t} M_{\boldsymbol{Y}_t} + M_{\boldsymbol{Y}_t} A_{\boldsymbol{U}_t}^\top - M_{\boldsymbol{Y}_t} S_{\boldsymbol{U}_t} M_{\boldsymbol{Y}_t} + \Sigma_{\boldsymbol{U}_t} \qquad \text{(reduced Riccati equation, } R \times R\text{)}$$

where $S = H^\top \Gamma^{-1} H$, $A_{\boldsymbol{U}_t} = \boldsymbol{U}_t^\top A \boldsymbol{U}_t$, $S_{\boldsymbol{U}_t} = \boldsymbol{U}_t^\top S \boldsymbol{U}_t$, $\Sigma_{\boldsymbol{U}_t} = \boldsymbol{U}_t^\top \Sigma \boldsymbol{U}_t$

- Practical way to solve moment equations in low-rank fashion
- Equations have already been proposed in [YO21] (and subsequent works) + connections to work in [SHNT23]
- (A symmetric) $\lim_{t\to\infty} \boldsymbol{U}_t = R$ dom. eigenmodes of A — connection to Assimilation in Unstable Subspace [BGA+17]

Accuracy of DLR-KBP approximation

- How do the dynamics of the DLR-KBP differ from that of KBP?

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\mathrm{KBP}} = (A\boldsymbol{\mathcal{X}}_t^{\mathrm{KBP}} + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\boldsymbol{\mathcal{W}}_t + P_t H^\top \Gamma^{-1}\left[\mathrm{d}\boldsymbol{\mathcal{Z}}_t - (H\boldsymbol{\mathcal{X}}_t^{\mathrm{KBP}}\mathrm{d}t + \Gamma^{1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t)\right]$$
$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\mathrm{DLR}} = (A\boldsymbol{\mathcal{X}}_t^{\mathrm{DLR}} + f)\mathrm{d}t + P_{\boldsymbol{U}_t}\Sigma^{1/2}\mathrm{d}\boldsymbol{\mathcal{W}}_t + P_t H^\top \Gamma^{-1}\left[\mathrm{d}\boldsymbol{\mathcal{Z}}_t - (H\boldsymbol{\mathcal{X}}_t^{\mathrm{DLR}}\mathrm{d}t + \Gamma^{1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t)\right],$$

- Deviation of DLR-KBP from KBP depends on the *orthogonal defect*

$$\|(I - \boldsymbol{U}_t\boldsymbol{U}_t^\top)\Sigma^{1/2}\|_F := \varepsilon_R(t)$$

Theorem : Mean and covariance bounds [NTT25]

Let $(m_t^{\mathrm{DLR}}, P_t^{\mathrm{DLR}})$ resp. $(m_t^{\mathrm{KBP}}, P_t^{\mathrm{KBP}})$ the (DLR-)KBP means and covariances. Then

$$\mathbb{E}\|m_T^{\mathrm{DLR}} - m_T^{\mathrm{KBP}}\|^2 + \|P_T^{\mathrm{DLR}} - P_T^{\mathrm{KBP}}\|_F^2 \le C(T)(\mathbb{E}\|m_0^{\mathrm{DLR}} - m_0^{\mathrm{KBP}}\|^2 + \|P_0^{\mathrm{KBP}} - P_0^{\mathrm{DLR}}\|_F^2 + \sup_{0\le t\le T}\varepsilon_R^2(t))$$

DLR-KBP approximation valid for

- ✗ moderate/large isotropic noise $\Sigma = \sigma I$ (e.g. unresolved model scales modelled uniformly)
- ✓ small/zero-noise/"perfect" models (e.g. idealised test regimes twin experiments, confidence in dynamics)
- ✓ low-rank noise $\Sigma = L_t L_t^\top$ (e.g. noise through low-dimensional forcing map)

Accuracy of DLR-KBP approximation

- How do the dynamics of the DLR-KBP differ from that of KBP?

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\mathrm{KBP}} = (A\boldsymbol{\mathcal{X}}_t^{\mathrm{KBP}} + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\boldsymbol{\mathcal{W}}_t + P_t H^\top \Gamma^{-1}\left[\mathrm{d}\boldsymbol{\mathcal{Z}}_t - (H\boldsymbol{\mathcal{X}}_t^{\mathrm{KBP}}\mathrm{d}t + \Gamma^{1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t)\right]$$
$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\mathrm{DLR}} = (A\boldsymbol{\mathcal{X}}_t^{\mathrm{DLR}} + f)\mathrm{d}t + P_{\boldsymbol{U}_t}\Sigma^{1/2}\mathrm{d}\boldsymbol{\mathcal{W}}_t + P_t H^\top \Gamma^{-1}\left[\mathrm{d}\boldsymbol{\mathcal{Z}}_t - (H\boldsymbol{\mathcal{X}}_t^{\mathrm{DLR}}\mathrm{d}t + \Gamma^{1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t)\right],$$

- Deviation of DLR-KBP from KBP depends on the *orthogonal defect*

$$\|(I - \boldsymbol{U}_t\boldsymbol{U}_t^\top)\Sigma^{1/2}\|_F \coloneqq \varepsilon_R(t)$$

Theorem : Mean and covariance bounds [NTT25]

Let $(m_t^{\mathrm{DLR}}, P_t^{\mathrm{DLR}})$ resp. $(m_t^{\mathrm{KBP}}, P_t^{\mathrm{KBP}})$ the (DLR-)KBP means and covariances. Then

$$\mathbb{E}\|m_T^{\mathrm{DLR}} - m_T^{\mathrm{KBP}}\|^2 + \|P_T^{\mathrm{DLR}} - P_T^{\mathrm{KBP}}\|_F^2 \le C(T)(\mathbb{E}\|m_0^{\mathrm{DLR}} - m_0^{\mathrm{KBP}}\|^2 + \|P_0^{\mathrm{KBP}} - P_0^{\mathrm{DLR}}\|_F^2 + \sup_{0\le t\le T}\varepsilon_R^2(t))$$

DLR-KBP approximation valid for

- ✗ moderate/large isotropic noise $\Sigma = \sigma I$ (e.g. unresolved model scales modelled uniformly)
- ✓ small/zero-noise/"perfect" models (e.g. idealised test regimes twin experiments, confidence in dynamics)
- ✓ low-rank noise $\Sigma = L_t L_t^\top$ (e.g. noise through low-dimensional forcing map)

DLR Ensemble-Kalman filter (DLR-ENKF)

- ENKF system is an Interacting Particle System version of KBP

$$
\begin{aligned}
\mathrm{d}\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} &= (A\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{W}}}_t^{(p)} + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)}\mathrm{d}t - \Gamma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{V}}}_t^{(p)}) \\
\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} &\sim \boldsymbol{\mathcal{N}}(m_0, P_0), \qquad\qquad p = 1, \dots, N
\end{aligned}
$$

$\rightsquigarrow$ Goal : develop a low-rank analog

Strategy : establish an Interacting Particle System version of DLR-KBP $\boldsymbol{\mathcal{X}}_t = U_t^0 + \boldsymbol{U}_t \boldsymbol{Y}_t^\top$

- replace $\boldsymbol{Y}_t = [Y_t^1, \dots, Y_t^R]$ by $\widehat{\boldsymbol{Y}}_t = \left[\widehat{Y}_t^1, \dots, \widehat{Y}_t^R\right]$ where $\widehat{Y}_t^i = [\widehat{Y}_t^{i,(1)}, \dots, \widehat{Y}_t^{i,(N)}]^\top$
- each particle $\widehat{\boldsymbol{Y}}_t^{(p)} = \left[\widehat{Y}_t^{1,(p)}, \dots, \widehat{Y}_t^{R,(p)}\right]$ has dimension R
- enforce $\mathbb{E}_N[\widehat{\boldsymbol{Y}}_t^i] := \frac{1}{N}\sum_{p=1}^N \widehat{Y}_t^{i,(p)} = 0$ (sample zero-mean)
- (re-)define $\widehat{\mathcal{X}}_t = \widehat{U}_t^0 + \boldsymbol{U}_t \widehat{\boldsymbol{Y}}_t^\top$
  - $\widehat{U}_t^0$ is the sample mean
  - $\widehat{P}_t = \frac{1}{N-1}\boldsymbol{U}_t \widehat{\boldsymbol{Y}}_t^\top \widehat{\boldsymbol{Y}}_t \boldsymbol{U}_t^\top$ sample covariance

DLR Ensemble-Kalman filter (DLR-ENKF)

- ENKF system is an Interacting Particle System version of KBP

$$
\begin{aligned}
\mathrm{d}\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} &= (A\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} + f)\mathrm{d}t + \Sigma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{W}}}_t^{(p)} + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)}\mathrm{d}t - \Gamma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{V}}}_t^{(p)}) \\
\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} &\sim \boldsymbol{\mathcal{N}}(m_0, P_0), \qquad\qquad p = 1, \dots, N
\end{aligned}
$$

$\rightsquigarrow$ Goal : develop a low-rank analog

Strategy : establish an Interacting Particle System version of DLR-KBP $\boldsymbol{\mathcal{X}}_t = U_t^0 + \boldsymbol{U}_t \boldsymbol{Y}_t^\top$

- replace $\boldsymbol{Y}_t = [Y_t^1, \dots, Y_t^R]$ by $\widehat{\boldsymbol{Y}}_t = \left[\widehat{Y}_t^1, \dots, \widehat{Y}_t^R\right]$ where $\widehat{Y}_t^i = [\widehat{Y}_t^{i,(1)}, \dots, \widehat{Y}_t^{i,(N)}]^\top$
- each particle $\widehat{\boldsymbol{Y}}_t^{(p)} = \left[\widehat{Y}_t^{1,(p)}, \dots, \widehat{Y}_t^{R,(p)}\right]$ has dimension R
- enforce $\mathbb{E}_N[\widehat{\boldsymbol{Y}}_t^i] := \frac{1}{N}\sum_{p=1}^N \widehat{Y}_t^{i,(p)} = 0$ (sample zero-mean)
- (re-)define $\widehat{\mathcal{X}}_t = \widehat{U}_t^0 + \boldsymbol{U}_t \widehat{\boldsymbol{Y}}_t^\top$
  - $\widehat{U}_t^0$ is the sample mean
  - $\widehat{P}_t = \frac{1}{N-1}\boldsymbol{U}_t \widehat{\boldsymbol{Y}}_t^\top \widehat{\boldsymbol{Y}}_t \boldsymbol{U}_t^\top$ sample covariance

We *propose* the following system

DLR-ENKF equations [NTT25]

$$
\begin{aligned}
\mathrm{d}\widehat{U}^0_t &= ((A - \widehat{P}_t S)\widehat{U}^0_t + f)\mathrm{d}t + \widehat{P}_t H^\top \Gamma^{-1}\mathrm{d}\boldsymbol{\mathcal{Z}}_t + P_{\boldsymbol{U}_t}\Sigma^{1/2}\mathrm{d}\mathbb{E}_P[\widehat{\mathcal{W}}_t] + \widehat{P}_t H^\top \Gamma^{-1/2}\mathrm{d}\mathbb{E}_N[\widehat{\boldsymbol{\mathcal{V}}}_t] \\
\mathrm{d}\boldsymbol{U}_t &= (I - \boldsymbol{U}_t\boldsymbol{U}_t^\top)A\boldsymbol{U}_t\mathrm{d}t \\
\mathrm{d}\widehat{\boldsymbol{Y}}_t^\top &= \boldsymbol{U}_t^\top(A - \widehat{P}_t S)\boldsymbol{U}_t\widehat{\boldsymbol{Y}}_t^\top \mathrm{d}t + \boldsymbol{U}_t^\top\Sigma^{1/2}\mathrm{d}\widehat{\mathcal{W}}^\star_t - \boldsymbol{U}_t^\top\widehat{P}_t H^\top\Gamma^{-1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{V}}}^\star_t
\end{aligned}
$$

where $S = H^\top\Gamma^{-1}H$ and $\widehat{\mathcal{W}}^\star_t = \widehat{\mathcal{W}}_t - \mathbb{E}_N[\widehat{\mathcal{W}}_t]$

Key property:

$$
\mathrm{d}\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} = (A\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)} + f)\mathrm{d}t + P_{\boldsymbol{U}_t}\Sigma^{1/2}\mathrm{d}\widehat{\mathcal{W}}_t^{(p)} + \widehat{P}_t H^\top\Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{\boldsymbol{\mathcal{X}}}_t^{(p)}\mathrm{d}t - \Gamma^{1/2}\mathrm{d}\widehat{\boldsymbol{\mathcal{V}}}_t^{(p)})
$$

✓ setup ensures $\mathbb{E}_N[\widehat{\boldsymbol{Y}}_t] = 0$ for $t \geq 0$

- $\widehat{U}^0_t$ is the sample mean
- $\boldsymbol{U}_t\widehat{\boldsymbol{Y}}_t^{(p)}$ is stochastic fluctuation of p-th particle

Propagation of chaos

- "validity" of ENKF formulation supported by convergence to KBP for $N \to \infty$

Theorem (ENKF propagation of chaos [DMT18])

Under assumptions, for $2(3n-1)d+1 \leq N$, it holds

$$\sup_{t\geq 0} \mathbb{E}[\|\widehat{\mathcal{X}}_t^{(1)} - \boldsymbol{\mathcal{X}}_t^{(1)}\|^n]^{\frac{1}{n}} < \frac{c_n}{\sqrt{N}}$$

- similarly, DLR-ENKF converges to DLR-KBP for $N \to \infty$

Theorem (DLR-ENKF propagation of chaos [NTT25])

Under assumptions, for $2(3n-1)R+1 \leq N$, it holds

$$\sup_{t\geq 0} \mathbb{E}[\|\widehat{\mathcal{X}}_t^{(1)} - \boldsymbol{\mathcal{X}}_t^{(1)}\|^n]^{\frac{1}{n}} < \frac{c_n}{\sqrt{N}}$$

where $\widehat{\mathcal{X}}_t^{(1)}$ the (DLR-)ENKF particle and $\boldsymbol{\mathcal{X}}_t^{(1)}$ (DLR-)KBP solution

Numerical experiment : RMSEs

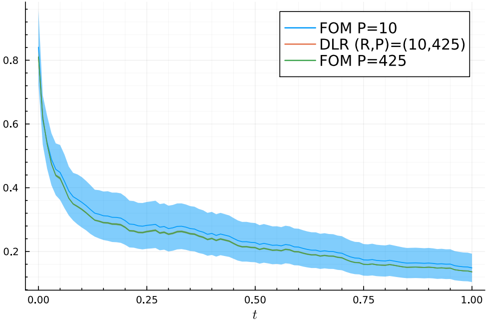


Figure: Full observations

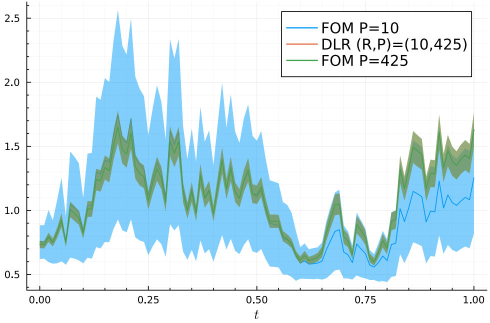


Figure: Partial observations

Figure: Mean RMSE $\pm$ standard deviation (10 runs): ENKF($N = 10$), ENKF($N = 425$) and DLR-ENKF($R = 10, N = 425$)

Table of Contents

Parameter Identification problems

State/parameter identification problem

Given known (deterministic) parametric dynamics $f : \mathbb{R}^d \times \mathbb{R}^{n_\theta} \to \mathbb{R}^d$ and linear observations

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\text{true}} = f(\boldsymbol{\mathcal{X}}_t^{\text{true}}, \theta^{\text{true}})\mathrm{d}t \qquad \mathrm{d}\boldsymbol{\mathcal{Z}}_t = H\boldsymbol{\mathcal{X}}_t^{\text{true}}\mathrm{d}t + \Gamma^{1/2}\mathrm{d}\widetilde{\boldsymbol{\mathcal{V}}}_t$$

where $(\boldsymbol{\mathcal{X}}_0^{\text{true}}, \theta^{\text{true}}) \sim \pi_{\text{pr}}$ but unknown, determine

$$\text{Law}(\boldsymbol{\mathcal{X}}_t^{\text{true}} \,|\, \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t}) \qquad \text{Law}(\theta^{\text{true}} \,|\, \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t})$$

Idea :

- Create augmented state $\bar{\boldsymbol{\mathcal{X}}}_t = [\boldsymbol{\mathcal{X}}_t, \theta_t]^\top \in \mathbb{R}^{d+n_\theta}$ with “fake dynamics” $\mathrm{d}\bar{\boldsymbol{\mathcal{X}}}_t = [f(\boldsymbol{\mathcal{X}}_t, \theta_t), 0]^\top \mathrm{d}t$
- Apply (Ensemble) Kalman filter equations

Augmented Kalman filter approach

$$\mathrm{d}\widehat{\mathcal{X}}_t^{(p)} = f(\widehat{\mathcal{X}}_t^{(p)}, \hat{\theta}_t^{(p)})\mathrm{d}t + \hat{P}_t^{xx} H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{\mathcal{X}}_t^{(p)}\mathrm{d}t - \Gamma^{1/2}\mathrm{d}\widehat{\mathcal{V}}_t^{(p)}) \qquad \text{(state)}$$

$$\mathrm{d}\hat{\theta}_t^{(p)} = \hat{P}_t^{\theta x} H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{\mathcal{X}}_t^{(p)}\mathrm{d}t - \Gamma^{1/2}\mathrm{d}\widehat{\mathcal{V}}_t^{(p)}) \qquad \text{(parameter)}$$

where $\hat{P}_t^{xx} = \frac{N}{N-1}\mathbb{E}_N[\widehat{\mathcal{X}}_t^\star(\widehat{\mathcal{X}}_t^\star)^\top]$, $\hat{P}_t^{\theta x} = \frac{N}{N-1}\mathbb{E}_N[\hat{\theta}_t^\star(\hat{\theta}_t^\star)^\top]$, $\hat{B}^\star = \hat{B} - \mathbb{E}_N[\hat{B}]$

DLR-ENKF for Parameter Identification

Goal : use DLR-ENKF to reduce computational costs

- assume n_θ moderate $\rightsquigarrow$ need only DLR reduction on state

Issues

1. theory developed only for linear-affine case
2. desire to use "robust" time-integration schemes
3. efficient handling of nonlinear terms

Solutions [NRTT26]

- $\rightsquigarrow$ consistent nonlinear DLR-ENKF formulation
- $\rightsquigarrow$ BUG + forecast/analysis scheme
- $\rightsquigarrow$ DEIM-based method

Numerical experiment: Fisher-KPP

The model :
Fisher-Kolmogorov-Petrovsky-Piskunov

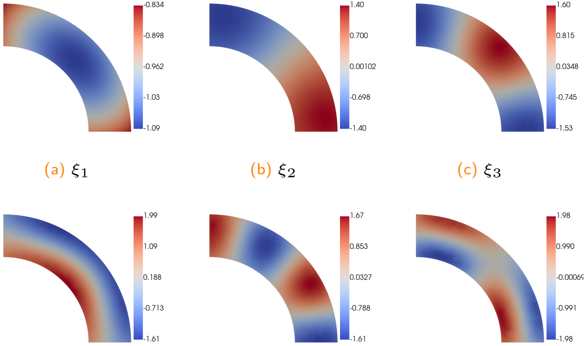
$$\partial_t u(t,x,\theta) = \mathrm{div}(\nu(x,\theta)\nabla u) + 75u(1-u)$$

where

$$\nu(x,\theta) = \sqrt{2} + \sum_{i=1}^{n_\theta} \theta_i \sqrt{\lambda_i}\xi_i(x)$$

for $\theta \in \mathbb{R}^{n_\theta}$ unknown ($n_\theta = 6$)

Aim : identify θ^{true}

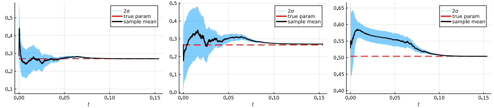


(a) ξ_1 (b) ξ_2 (c) ξ_3

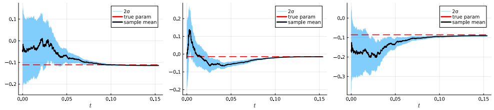
(d) ξ_4 (e) ξ_5 (f) ξ_6

Figure: Modes of the diffusion coefficient $\nu(x)$

Fisher-KPP : results

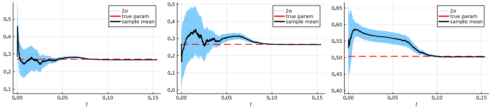


(a) θ_1 (b) θ_2 (c) θ_3

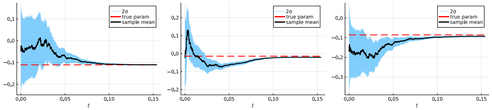


(d) θ_4 (e) θ_5 (f) θ_6

Figure: ENKF parameter identification



(a) θ_1 (b) θ_2 (c) θ_3



(d) θ_4 (e) θ_5 (f) θ_6

Figure: DLR-ENKF ($R = 7$) parameter identification

Discretisation parameters : $|\boldsymbol{\mathcal{X}}_t| = 540$, $N = 200$, 3500 time-steps

Conclusion

In this work, we

- ✓ developed a Data Assimilation-compatible DLR framework
- ✓ derived DLR-KBP equations
  - reduced moment equations, well-posedness, Oja flow, moment error bounds
- ✓ proposed DLR-ENKF system
  - well-posedness, propagation of chaos
- ✓ extended the DLR-ENKF to parameter identification problems
  - formulations, time-integrators, hyperreduction, rank-adaptivity, numerical tests

Future work

- explore integration of DLR frameworks in particle filters

Thank you for your attention !
Questions ?

References

[BGA+17] Marc Bocquet, Karthik S. Gurumoorthy, Amit Apte, Alberto Carrassi, Colin Grudzien, and Christopher K. R. T. Jones, *Degenerate Kalman Filter Error Covariances and Their Convergence onto the Unstable Subspace*, SIAM/ASA Journal on Uncertainty Quantification **5** (2017), 304–333 (en).

[CKL22] Gianluca Ceruti, Jonas Kusch, and Christian Lubich, *A rank-adaptive robust integrator for dynamical low-rank approximation*, BIT Numerical Mathematics **62** (2022), 1149–1174 (en).

[DMT18] P. Del Moral and J. Tugaut, *On the stability and the uniform propagation of chaos properties of Ensemble Kalman–Bucy filters*, The Annals of Applied Probability **28** (2018).

[DPN+23] M. Donello, G. Palkar, M. H. Naderi, D. C. Del Rey Ferná ndez, and H. Babaee, *Oblique projection for scalable rank-adaptive reduced-order modelling of nonlinear stochastic partial differential equations with time-dependent bases*, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences **479** (2023) (en).

[KNZ25] Yoshihito Kazashi, Fabio Nobile, and Fabio Zoccolan, *Dynamical low-rank approximation for stochastic differential equations*, Mathematics of Computation **94** (2025), no. 353, 1335–1375.

[NRTT26] Fabio Nobile, Sébastien Riffaud, and Thomas Trigo Trindade, *Dynamical Low-Rank Ensemble Kalman filter for State/ Parameter estimation*, preprint arXiv:2602.06614 (2026).

[NTT25] Fabio Nobile and Thomas Trigo Trindade, *Dynamical Low-Rank Approximations for Kalman Filtering*, preprint arXiv:2509.11210 (2025).

[SHNT23] Jonathan Schmidt, Philipp Hennig, Jörg Nick, and Filip Tronarp, *The rank-reduced Kalman filter: Approximate dynamical-low-rank filtering in high dimensions*, Advances in Neural Information Processing Systems **36** (2023), 61364–61376.

[YO21] Shuto Yamada and Kentaro Ohki, *On a new low-rank Kalman-Bucy filter and its convergence property*, Proceedings of the ISCIE International Symposium on Stochastic Systems Theory and its Applications, vol. 2021, The ISCIE Symposium on Stochastic Systems Theory and Its Applications, 2021, pp. 16–20.

DLR Kalman-Bucy Process (DLR-KBP)

Goals

- derive a *principled* DLR-type reduction for Kalman-Bucy Processes
- extend to Ensemble Kalman systems

Strategy

- Step 1: *Generalise* DLR-SDE formulation [KNZ25] to processes with additional Itô process, i.e.

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\mathrm{true}} = a(t,\omega)\mathrm{d}t + b(t,\omega)\mathrm{d}\overline{\boldsymbol{\mathcal{W}}}_t + c(t,\omega)\mathrm{d}\boldsymbol{\mathcal{Z}}_t$$

- (Wishlist) We want a process $\boldsymbol{\mathcal{X}}_t^{\mathrm{DLR}} \approx \boldsymbol{\mathcal{X}}_t^{\mathrm{true}}$ verifying the following properties
  1. (Low-rank decomposition)

$$\boldsymbol{\mathcal{X}}_t^{\mathrm{DLR}} = U_t^0 + \sum_{i=1}^{R} U_t^i Y_t^i = U_t^0 + \boldsymbol{U}_t \boldsymbol{Y}_t^\top \quad \forall t \geq 0$$

  2. (Modes characterisation)

$$\mathrm{d}U_t^0 = \alpha_t^0 \mathrm{d}t + \beta_t^0 \mathrm{d}\boldsymbol{\mathcal{Z}}_t \qquad \mathrm{d}U_t^i = \alpha_t^i \mathrm{d}t + \beta_t^i \mathrm{d}\boldsymbol{\mathcal{Z}}_t \qquad \mathrm{d}Y_t^i = \gamma_t^i \mathrm{d}t + \delta_t^i \mathrm{d}\overline{\boldsymbol{\mathcal{W}}}_t + \varepsilon_t^i \mathrm{d}\boldsymbol{\mathcal{Z}}_t$$

  3. (Physical modes constraint)

$$(U_t^i)^\top \alpha_t^j = 0 \qquad (U_t^i)^\top \beta_t^j = 0 \qquad 1 \leq i,j \leq R$$

DLR Kalman-Bucy Process (DLR-KBP)

Goals

- derive a *principled* DLR-type reduction for Kalman-Bucy Processes
- extend to Ensemble Kalman systems

Strategy

- Step 1: *Generalise* DLR-SDE formulation [KNZ25] to processes with additional Itô process, i.e.

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\text{true}} = a(t,\omega)\mathrm{d}t + b(t,\omega)\mathrm{d}\overline{\boldsymbol{\mathcal{W}}}_t + c(t,\omega)\mathrm{d}\boldsymbol{\mathcal{Z}}_t$$

  - (Wishlist) We want a process $\boldsymbol{\mathcal{X}}_t^{\text{DLR}} \approx \boldsymbol{\mathcal{X}}_t^{\text{true}}$ verifying the following properties
    1. (Low-rank decomposition)

$$\boldsymbol{\mathcal{X}}_t^{\text{DLR}} = U_t^0 + \sum_{i=1}^{R} U_t^i Y_t^i = U_t^0 + \boldsymbol{U}_t \boldsymbol{Y}_t^\top \quad \forall t \geq 0$$

    2. (Modes characterisation)

$$\mathrm{d}U_t^0 = \alpha_t^0 \mathrm{d}t + \beta_t^0 \mathrm{d}\boldsymbol{\mathcal{Z}}_t \qquad \mathrm{d}U_t^i = \alpha_t^i \mathrm{d}t + \beta_t^i \mathrm{d}\boldsymbol{\mathcal{Z}}_t \qquad \mathrm{d}Y_t^i = \gamma_t^i \mathrm{d}t + \delta_t^i \mathrm{d}\overline{\boldsymbol{\mathcal{W}}}_t + \varepsilon_t^i \mathrm{d}\boldsymbol{\mathcal{Z}}_t$$

    3. (Physical modes constraint)

$$\left(U_t^i\right)^\top \alpha_t^j = 0 \qquad \left(U_t^i\right)^\top \beta_t^j = 0 \qquad 1 \leq i,j \leq R$$

DLR Kalman-Bucy Process (DLR-KBP)

Goals

- derive a *principled* DLR-type reduction for Kalman-Bucy Processes
- extend to Ensemble Kalman systems

Strategy

- Step 1: *Generalise* DLR-SDE formulation [KNZ25] to processes with additional Itô process, i.e.

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\text{true}} = a(t,\omega)\mathrm{d}t + b(t,\omega)\mathrm{d}\overline{\boldsymbol{\mathcal{W}}}_t + c(t,\omega)\mathrm{d}\boldsymbol{\mathcal{Z}}_t$$

  - (Wishlist) We want a process $\boldsymbol{\mathcal{X}}_t^{\text{DLR}} \approx \boldsymbol{\mathcal{X}}_t^{\text{true}}$ verifying the following properties
    1. (Low-rank decomposition)

$$\boldsymbol{\mathcal{X}}_t^{\text{DLR}} = U_t^0 + \sum_{i=1}^{R} U_t^i Y_t^i = U_t^0 + \boldsymbol{U}_t \boldsymbol{Y}_t^\top \quad \forall t \geq 0$$

    2. (Modes characterisation)

$$\mathrm{d}U_t^0 = \alpha_t^0 \mathrm{d}t + \beta_t^0 \mathrm{d}\boldsymbol{\mathcal{Z}}_t \qquad \mathrm{d}U_t^i = \alpha_t^i \mathrm{d}t + \beta_t^i \mathrm{d}\boldsymbol{\mathcal{Z}}_t \qquad \mathrm{d}Y_t^i = \gamma_t^i \mathrm{d}t + \delta_t^i \mathrm{d}\overline{\boldsymbol{\mathcal{W}}}_t + \varepsilon_t^i \mathrm{d}\boldsymbol{\mathcal{Z}}_t$$

    3. (Physical modes constraint)

$$(U_t^i)^\top \alpha_t^j = 0 \qquad (U_t^i)^\top \beta_t^j = 0 \qquad 1 \leq i,j \leq R$$

DLR Kalman-Bucy Process (DLR-KBP)

Goals

- derive a *principled* DLR-type reduction for Kalman-Bucy Processes
- extend to Ensemble Kalman systems

Strategy

- Step 1: *Generalise* DLR-SDE formulation [KNZ25] to processes with additional Itô process, i.e.

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\text{true}} = a(t,\omega)\mathrm{d}t + b(t,\omega)\mathrm{d}\overline{\boldsymbol{\mathcal{W}}}_t + c(t,\omega)\mathrm{d}\boldsymbol{\mathcal{Z}}_t$$

  - (Wishlist) We want a process $\boldsymbol{\mathcal{X}}_t^{\text{DLR}} \approx \boldsymbol{\mathcal{X}}_t^{\text{true}}$ verifying the following properties
    1. (Low-rank decomposition)

$$\boldsymbol{\mathcal{X}}_t^{\text{DLR}} = U_t^0 + \sum_{i=1}^{R} U_t^i Y_t^i = U_t^0 + \boldsymbol{U}_t \boldsymbol{Y}_t^\top \quad \forall t \geq 0$$

    2. (Modes characterisation)

$$\mathrm{d}U_t^0 = \alpha_t^0 \mathrm{d}t + \beta_t^0 \mathrm{d}\boldsymbol{\mathcal{Z}}_t \qquad \mathrm{d}U_t^i = \alpha_t^i \mathrm{d}t + \beta_t^i \mathrm{d}\boldsymbol{\mathcal{Z}}_t \qquad \mathrm{d}Y_t^i = \gamma_t^i \mathrm{d}t + \delta_t^i \mathrm{d}\overline{\boldsymbol{\mathcal{W}}}_t + \varepsilon_t^i \mathrm{d}\boldsymbol{\mathcal{Z}}_t$$

    3. (Physical modes constraint)

$$(U_t^i)^\top \alpha_t^j = 0 \qquad (U_t^i)^\top \beta_t^j = 0 \qquad 1 \leq i,j \leq R$$

We *propose* the following system

Mean-separated DLR equations

$$\mathrm{d}U_t^0 = \mathbb{E}[a \mid \mathcal{A}_{\mathcal{Z}_t}]\mathrm{d}t + \mathbb{E}[c \mid \mathcal{A}_{\mathcal{Z}_t}]\mathrm{d}\mathcal{Z}_t$$

$$\mathrm{d}\boldsymbol{U}_t = P_{\boldsymbol{U}_t}^{\perp}\mathbb{E}\left[(a^\star - G_t)(\boldsymbol{Y}_t M_{\boldsymbol{Y}_t}^{-1}) \mid \mathcal{A}_{\mathcal{Z}_t}\right]\mathrm{d}t + P_{\boldsymbol{U}_t}^{\perp}\mathbb{E}[c^\star(\boldsymbol{Y}_t M_{\boldsymbol{Y}_t}^{-1}) \mid \mathcal{A}_{\mathcal{Z}_t}]\mathrm{d}\mathcal{Z}_t$$

$$\mathrm{d}\boldsymbol{Y}_t^\top = M_{\boldsymbol{U}_t}^{-1}\boldsymbol{U}_t^\top a^\star \mathrm{d}t + M_{\boldsymbol{U}_t}^{-1}\boldsymbol{U}_t^\top b\mathrm{d}\overline{\mathcal{W}}_t + M_{\boldsymbol{U}_t}^{-1}\boldsymbol{U}_t^\top c^\star \mathrm{d}\mathcal{Z}_t$$

where $f^\star = f - \mathbb{E}[f \mid \mathcal{A}_{\mathcal{Z}_t}]$, $M_{\boldsymbol{Y}_t} = \mathbb{E}[\boldsymbol{Y}_t^\top \boldsymbol{Y}_t] \in \mathrm{Mat}(R \times R)$, $M_{\boldsymbol{U}_t} = \boldsymbol{U}_t^\top \boldsymbol{U}_t \in \mathrm{Mat}(R \times R)$, $P_{\boldsymbol{U}_t} = \boldsymbol{U}_t M_{\boldsymbol{U}_t}^{-1}\boldsymbol{U}_t^\top$, $P_{\boldsymbol{U}_t}^{\perp} = I - P_{\boldsymbol{U}_t}$ and

$$G_t = \sum_{i=1}^{R}\sum_{s,s'=1}^{k}\left[\Pi_{\boldsymbol{U}_t}^{\perp}\mathbb{E}[c_t^\star(\boldsymbol{Y}_t M_{\boldsymbol{Y}_t}^{-1})_i \mid \mathcal{A}_{\mathcal{Z}_t}\right](:,s) \cdot \left[(M_{\boldsymbol{U}_t}^{-1}\boldsymbol{U}_t)_i^\top c_t^\star\right](s') \cdot \mathrm{d}[\mathcal{Z}_t^s, \mathcal{Z}_t^{s'}]_t$$

✓ verifies all requirements in previous slide

Theorem (DLR system consistency, simplified) [NTT25]

If $\mathcal{X}_t^{\mathrm{true}} = \tilde{U}_t^0 + \sum_{i=1}^{R}\tilde{U}_t^i\tilde{Y}_t^i$ for some $R \geq 1$ and $t \geq 0$, then under technical conditions, the triplet $(\tilde{U}_t^0, \tilde{U}_t^j, \tilde{Y}_t^j)$ solve the DLR equations

✓ "if $\mathcal{X}_t^{\mathrm{true}}$ were exactly low-rank, then it would satisfy the DLR equations"

We *propose* the following system

Mean-separated DLR equations

$$\mathrm{d}U_t^0 = \mathbb{E}[a \mid \mathcal{A}_{\mathcal{Z}_t}]\mathrm{d}t + \mathbb{E}[c \mid \mathcal{A}_{\mathcal{Z}_t}]\mathrm{d}\mathcal{Z}_t$$

$$\mathrm{d}\boldsymbol{U}_t = P^{\perp}_{\boldsymbol{U}_t}\mathbb{E}\left[(a^\star - G_t)(\boldsymbol{Y}_t M^{-1}_{\boldsymbol{Y}_t}) \mid \mathcal{A}_{\mathcal{Z}_t}\right]\mathrm{d}t + P^{\perp}_{\boldsymbol{U}_t}\mathbb{E}[c^\star(\boldsymbol{Y}_t M^{-1}_{\boldsymbol{Y}_t}) \mid \mathcal{A}_{\mathcal{Z}_t}]\mathrm{d}\mathcal{Z}_t$$

$$\mathrm{d}\boldsymbol{Y}_t^\top = M^{-1}_{\boldsymbol{U}_t}\boldsymbol{U}_t^\top a^\star \mathrm{d}t + M^{-1}_{\boldsymbol{U}_t}\boldsymbol{U}_t^\top b\mathrm{d}\overline{\mathcal{W}}_t + M^{-1}_{\boldsymbol{U}_t}\boldsymbol{U}_t^\top c^\star \mathrm{d}\mathcal{Z}_t$$

where $f^\star = f - \mathbb{E}[f \mid \mathcal{A}_{\mathcal{Z}_t}]$, $M_{\boldsymbol{Y}_t} = \mathbb{E}[\boldsymbol{Y}_t^\top \boldsymbol{Y}_t] \in \mathrm{Mat}(R \times R)$, $M_{\boldsymbol{U}_t} = \boldsymbol{U}_t^\top \boldsymbol{U}_t \in \mathrm{Mat}(R \times R)$, $P_{\boldsymbol{U}_t} = \boldsymbol{U}_t M^{-1}_{\boldsymbol{U}_t}\boldsymbol{U}_t^\top$, $P^{\perp}_{\boldsymbol{U}_t} = I - P_{\boldsymbol{U}_t}$ and

$$G_t = \sum_{i=1}^{R}\sum_{s,s'=1}^{k}\left[\Pi^{\perp}_{\boldsymbol{U}_t}\mathbb{E}[c_t^\star(\boldsymbol{Y}_t M^{-1}_{\boldsymbol{Y}_t})_i \mid \mathcal{A}_{\mathcal{Z}_t}\right](:,s)\cdot\left[(M^{-1}_{\boldsymbol{U}_t}\boldsymbol{U}_t)_i^\top c_t^\star\right](s')\cdot \mathrm{d}[\mathcal{Z}_t^s, \mathcal{Z}_t^{s'}]_t$$

✓ verifies all requirements in previous slide

Theorem (DLR system consistency, simplified) [NTT25]

If $\mathcal{X}_t^{\text{true}} = \tilde{U}_t^0 + \sum_{i=1}^{R}\tilde{U}_t^i\tilde{Y}_t^i$ for some $R \geq 1$ and $t \geq 0$, then under technical conditions, the triplet $(\tilde{U}_t^0, \tilde{U}_t^j, \tilde{Y}_t^j)$ solve the DLR equations

✓ "if $\mathcal{X}_t^{\text{true}}$ were exactly low-rank, then it would satisfy the DLR equations"

Numerical experiment 1 : (toy) linear advection-reaction model

- Signal : $\mathrm{d}\boldsymbol{\mathcal{X}}_t^{\text{signal}} = (A\boldsymbol{\mathcal{X}}_t^{\text{signal}} + f)\mathrm{d}t + \sigma^{1/2}\mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_t$
  - $d = 100$
  - A and f upwind finite difference discretisation of $\partial_t u = -\partial_x u - 10^{-3}u + f$
- Observations : $\mathrm{d}\boldsymbol{\mathcal{Z}}_t = \boldsymbol{\mathcal{X}}_t^{\text{signal}}\mathrm{d}t + \gamma^{1/2}\mathrm{d}\widetilde{\boldsymbol{\mathcal{V}}}_t$
  - full observations, $k = d = 100$
  - $\gamma = 2$
- Initial condition : $u_0 = \sin(\frac{2\pi x}{L}) + \sum_{k=1}^{R_{\text{true}}} \frac{1}{i}\sin(\frac{2\pi x L}{k})\xi_k = U_0^0 + \boldsymbol{U}_0\boldsymbol{Y}_0^\top$
  - $\xi_k \overset{i.i.d}{\sim} \boldsymbol{\mathcal{N}}(0,1)$
  - *initial rank* $R_0 = 25$
- Time-stepping : Euler-Maruyama, $\Delta t = 10^{-4}$

Mean + covariance deviations : fixed rank $R = 15$, varying $\Sigma = \sigma I$

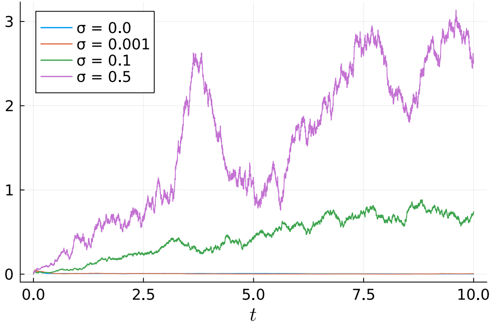


Figure: Errors $\|m_t^{\mathrm{DLR}} - m_t^{\mathrm{KBP}}\|$

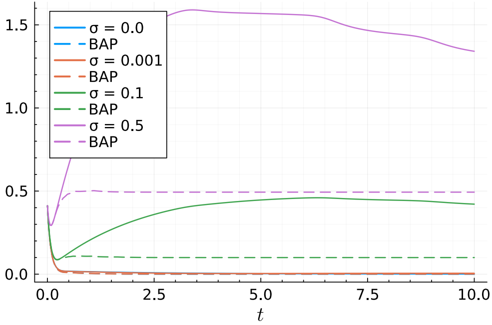


Figure: Errors $\|P_t^{\mathrm{DLR}} - P_t^{\mathrm{KBP}}\|_F$

Figure: Mean and covariance errors between KBP and DLR-KBP. The DLR-KBP rank is $R = 15$. BAP = Best Approximation Possible $= \|P_t^{\mathrm{KBP}} - \boldsymbol{\mathcal{T}}_R(P_t^{\mathrm{KBP}})\|_F$.

Mean + covariance deviations : varying rank R, fixed $\Sigma = 10^{-3} I$

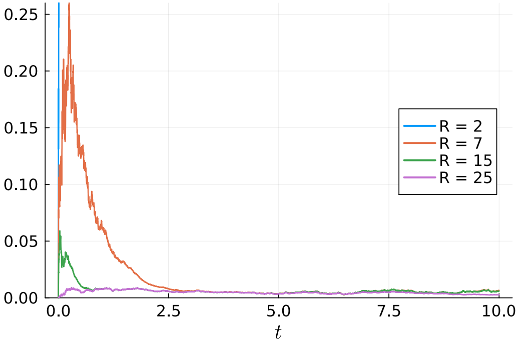


Figure: Errors $\|m_t^{\mathrm{DLR}}(R) - m_t^{\mathrm{KBP}}\|$

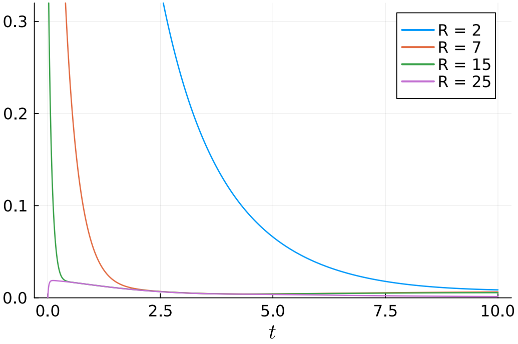


Figure: Errors $\|P_t^{\mathrm{DLR}}(R) - P_t^{\mathrm{KBP}}\|_F$

Figure: Mean and covariance errors between KBP and DLR-KBP.

Signal-tracking (RMSE) metrics

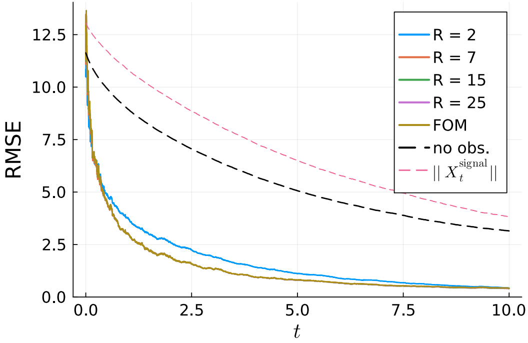


Figure: RMSE

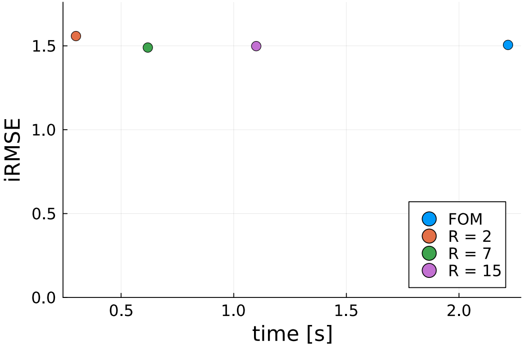


Figure: Integrated RMSEs v. computational cost in [s]

$$\mathrm{RMSE}(m_t, P_t) = \sqrt{\|m_t - \boldsymbol{\mathcal{X}}_t^{\text{signal}}\|^2 + \mathrm{tr}(P_t)}$$

$$\mathrm{iRMSE}(m_t, P_t) = \frac{\Delta t}{T} \sum_{i=1}^{N} \mathrm{RMSE}(m_{t_i}, P_{t_i})$$

Numerical experiment 2 : air pollution model (advection-diffusion)

- Signal : $2D$ FEM discretisation of SPDE $\mathrm{d}x_t^{\text{signal}} = (a\Delta x_t^{\text{signal}} + \boldsymbol{b} \cdot \nabla x_t^{\text{signal}})\mathrm{d}t + \sigma^{1/2}\mathrm{d}\widetilde{W}_t$
  - $d = 420$
  - semi-implicit time-stepping scheme
  - $\sigma = 10^{-5}$
- Observations : full/partial observations
  - full observations : $k = d = 420$
  - partial observations : $k = 25$ (linear functionals)
  - $\gamma = 10^{-2}$
- Initial condition : $u_0(x) = \exp(-(x_1 - 0.5)^2 - (x_2 - 0.5)^2) + \sum_{i=1}^{R_{\mathrm{true}}} \frac{1}{i^2} \sin(i\pi x_1)\cos(i\pi x_2)\xi_i$
  - $\xi_k \overset{i.i.d}{\sim} \mathcal{N}(0,1)$
  - *true rank* $R_{\text{true}} = 12$
- Time-stepping : Euler-Maruyama, $\Delta t = 10^{-4}$

Partial observations + accuracy of time-stepping scheme

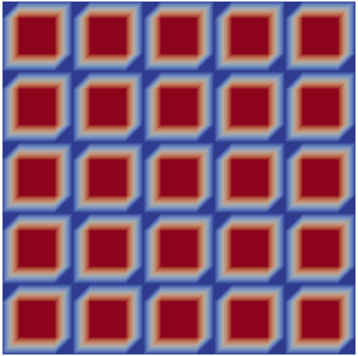


Figure: Each indicator function χ_ℓ is associated to one of the red squares.

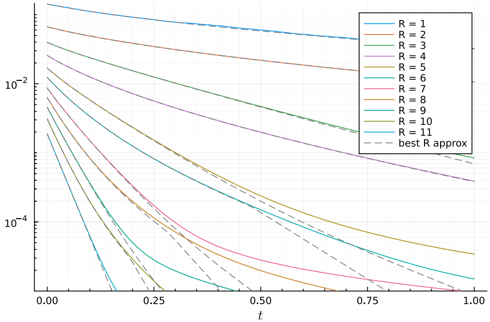


Figure: Errors $\|\widehat{\mathcal{X}}_t^{\mathrm{ENKF}} - \widehat{\mathcal{X}}_t^{\mathrm{DLR-ENKF}}\|_{\ell^2_P(H)}$.

Time-integrator

1. Mean update

$$(M - \Delta t A)U^0_{n+1} = MU^0_n + \sigma^{1/2} M P_{\boldsymbol{U}_n} \mathbb{E}_N[\Delta \widehat{\mathcal{W}}_n] + M\widehat{P}_t H^\top \Gamma^{-1}\left[\Delta \boldsymbol{\mathcal{Z}}_n - HU^0_n \Delta t + \Gamma^{1/2}\mathbb{E}_P[\Delta \widehat{\boldsymbol{\mathcal{V}}}_n]\right]$$

2. Physical modes update.

$$(M - \Delta t A)\widetilde{\boldsymbol{U}}_{n+1} = M\boldsymbol{U}_n - M\widehat{P}_t S \boldsymbol{U}_n \Delta t$$

3. Orthonormalise $\overline{\boldsymbol{U}}_{n+1} = \mathtt{qr}([\boldsymbol{U}_n, \widetilde{\boldsymbol{U}}_{n+1}])$ such that $\overline{\boldsymbol{U}}^\top_{n+1} M \overline{\boldsymbol{U}}_{n+1} = \boldsymbol{I}_{2R}$.
4. Stochastic mode update. Define $\widetilde{\boldsymbol{Y}}^\top_n = \overline{\boldsymbol{U}}^\top_{n+1} M \boldsymbol{U}_n \widehat{\boldsymbol{Y}}^\top_n$ and solve

$$(I - \Delta t \overline{\boldsymbol{U}}^\top_{n+1} A \overline{\boldsymbol{U}}_{n+1})\widetilde{\boldsymbol{Y}}^\top_{n+1} = \widetilde{\boldsymbol{Y}}^\top_n - \overline{\boldsymbol{U}}^\top_{n+1} M \widehat{P}_n S \boldsymbol{\mathcal{X}}^\star_n \Delta t + \overline{\boldsymbol{U}}^\top_{n+1} \Sigma^{1/2} \Delta \boldsymbol{\mathcal{W}}^\star_n - \overline{\boldsymbol{U}}^\top_{n+1} M \widehat{P}_n H^\top \Gamma^{-1/2} \Delta \boldsymbol{\mathcal{V}}^\star_n.$$

5. Truncation

$$\mathcal{T}_R(\widetilde{\boldsymbol{Y}}_{n+1}) = U_R S_R V_R^\top \qquad \boldsymbol{U}_{n+1} = \overline{\boldsymbol{U}}_{n+1} U_R \qquad \widehat{\boldsymbol{Y}}_{n+1} = V_R S_R.$$

(where $\mathbb{E}_N[V_R^\top V_R] = \boldsymbol{I}_{2R}$.)

Numerical experiment 1 : propagation of chaos

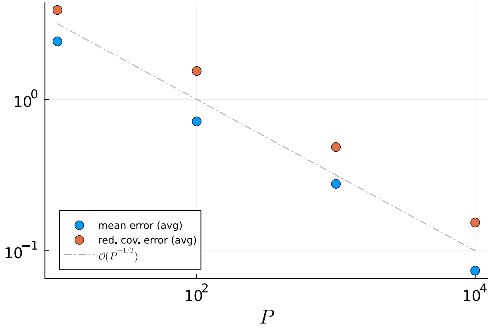


Figure: Covariance + mean MC error

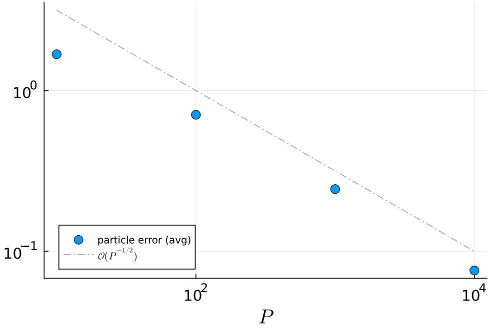


Figure: Propagation of chaos

Figure: DLR-ENKF approximation error of DLR-KBP with N particles

DLR-ENKF well-posedness

- Two necessary assumptions

Full observations

$$H^\top \Gamma^{-1} H = S = \rho \boldsymbol{I}_d \quad \rho > 0 \qquad \text{(A1)}$$

Dissipative dynamics

$$\lambda_{\max}(A + A^\top) < 0 \qquad \text{(A2)}$$

Theorem (Strong solution of DLR-ENKF) [NTT25]

Assuming (A1) and (A2) hold true and that the number of particles verifies $N > 4R - 1$.
Then the DLR-ENKF system is well-posed.

- proof relies on establishing non-explosion of paths
- dissipative dynamics are used mainly to control moments of DLR-ENKF covariance

DLR-ENKF well-posedness

- Two necessary assumptions

Full observations

$$H^\top \Gamma^{-1} H = S = \rho \boldsymbol{I}_d \quad \rho > 0 \tag{A1}$$

Dissipative dynamics

$$\lambda_{\max}(A + A^\top) < 0 \tag{A2}$$

Theorem (Strong solution of DLR-ENKF) [NTT25]

Assuming (A1) and (A2) hold true and that the number of particles verifies $N > 4R - 1$.
Then the DLR-ENKF system is well-posed.

- proof relies on establishing non-explosion of paths
- dissipative dynamics are used mainly to control moments of DLR-ENKF covariance

Time-continuous DLR-ENKF equations

Pipeline: nonlinear equations ⇝ apply DLR-SDE equations ⇝ used to set up Interacting Particle System

DLR Vanilla KBP

$$\mathrm{d}U_t^0 = \mathbb{E}[f(\boldsymbol{\mathcal{X}}_t,\theta_t)\,|\,\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t}]\mathrm{d}t + P_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0\mathrm{d}t)$$

$$\mathrm{d}\boldsymbol{U}_t = P_{\boldsymbol{U}_t}^{\perp}\mathbb{E}[f_t^\star(\boldsymbol{\mathcal{X}}_t,\theta_t)(\boldsymbol{Y}_t M_{\boldsymbol{Y}_t}^{-1})\,|\,\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t}]\mathrm{d}t$$

$$\mathrm{d}\boldsymbol{Y}_t^\top = \boldsymbol{U}_t^\top f_t^\star(\boldsymbol{\mathcal{X}}_t,\theta_t)\mathrm{d}t - \boldsymbol{U}_t^\top P_t H^\top \Gamma^{-1}(H\boldsymbol{\mathcal{X}}_t^\star + \Gamma^{1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t)$$

DLR Deterministic KBP

$$\mathrm{d}\widehat{U}_t^0 = \mathbb{E}[f(\widehat{\mathcal{X}}_t,\hat{\theta}_t)]\mathrm{d}t + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0\mathrm{d}t)$$

$$\mathrm{d}\widehat{\boldsymbol{U}}_t = P_{\widehat{\boldsymbol{U}}_t}^{\perp}\mathbb{E}[f_t^\star(\widehat{\mathcal{X}}_t,\hat{\theta}_t)\widehat{\boldsymbol{Y}}_t]\widehat{M}_{\widehat{\boldsymbol{Y}}_t}^{-1}\mathrm{d}t$$

$$\mathrm{d}\widehat{\boldsymbol{Y}}_t^\top = \widehat{\boldsymbol{U}}_t^\top f_t^\star(\widehat{\mathcal{X}}_t,\hat{\theta}_t)\mathrm{d}t - \frac{1}{2}\widehat{\boldsymbol{U}}_t^\top \widehat{P}_t H^\top \Gamma^{-1} H\widehat{\mathcal{X}}_t^\star\mathrm{d}t$$

DLR-VENKF

$$\mathrm{d}\widehat{U}_t^0 = \mathbb{E}_N[f(\widehat{\mathcal{X}}_t,\hat{\theta}_t)]\mathrm{d}t + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{U}_t^0\mathrm{d}t - \Gamma^{1/2}\mathrm{d}\mathbb{E}_N[\widehat{\boldsymbol{\mathcal{V}}}_t])$$

$$\mathrm{d}\widehat{\boldsymbol{U}}_t = P_{\widehat{\boldsymbol{U}}_t}^{\perp}\mathbb{E}_N[f_t^\star(\widehat{\mathcal{X}}_t,\hat{\theta}_t)\widehat{\boldsymbol{Y}}_t]\widehat{M}_{\widehat{\boldsymbol{Y}}_t}^{-1}\mathrm{d}t$$

$$\mathrm{d}\widehat{\boldsymbol{Y}}_t^\top = \widehat{\boldsymbol{U}}_t^\top f_t^\star(\widehat{\mathcal{X}}_t,\hat{\theta}_t)\mathrm{d}t - \widehat{\boldsymbol{U}}_t^\top \widehat{P}_t H^\top \Gamma^{-1}(H\widehat{\mathcal{X}}_t^\star + \Gamma^{1/2}\mathrm{d}\widehat{\mathcal{V}}_t^\star)$$

DLR-DENKF

$$\mathrm{d}\widehat{U}_t^0 = \mathbb{E}_N[f(\widehat{\mathcal{X}}_t,\hat{\theta}_t)]\mathrm{d}t + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0\mathrm{d}t)$$

$$\mathrm{d}\widehat{\boldsymbol{U}}_t = P_{\widehat{\boldsymbol{U}}_t}^{\perp}\mathbb{E}_N[f_t^\star(\widehat{\mathcal{X}}_t,\hat{\theta}_t)\widehat{\boldsymbol{Y}}_t]\widehat{M}_{\widehat{\boldsymbol{Y}}_t}^{-1}\mathrm{d}t$$

$$\mathrm{d}\widehat{\boldsymbol{Y}}_t^\top = \widehat{\boldsymbol{U}}_t^\top f_t^\star(\widehat{\mathcal{X}}_t,\hat{\theta}_t)\mathrm{d}t - \frac{1}{2}\widehat{\boldsymbol{U}}_t^\top \widehat{P}_t H^\top \Gamma^{-1} H\widehat{\mathcal{X}}_t^\star\mathrm{d}t$$

We consider 3 ENKF variants: Vanilla (VENKF), Deterministic (DENKF), “Sequential” (SENKF)

SENKF = VENKF without $\Gamma^{1/2}\mathrm{d}\widehat{\mathcal{V}}_t$ term

Time-continuous DLR-ENKF equations

Pipeline: nonlinear equations $\rightsquigarrow$ apply DLR-SDE equations $\rightsquigarrow$ used to set up Interacting Particle System

DLR Vanilla KBP

$$
\begin{aligned}
\mathrm{d}U_t^0 &= \mathbb{E}[f(\boldsymbol{\mathcal{X}}_t,\theta_t)\,|\,\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t}]\mathrm{d}t + P_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0\mathrm{d}t)\\
\mathrm{d}\boldsymbol{U}_t &= P^\perp_{\boldsymbol{U}_t}\mathbb{E}[f_t^\star(\boldsymbol{\mathcal{X}}_t,\theta_t)(\boldsymbol{Y}_t M_{\boldsymbol{Y}_t}^{-1})\,|\,\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t}]\mathrm{d}t\\
\mathrm{d}\boldsymbol{Y}_t^\top &= \boldsymbol{U}_t^\top f_t^\star(\boldsymbol{\mathcal{X}}_t,\theta_t)\mathrm{d}t - \boldsymbol{U}_t^\top P_t H^\top\Gamma^{-1}(H\boldsymbol{\mathcal{X}}_t^\star + \Gamma^{1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t)
\end{aligned}
$$

DLR Deterministic KBP

$$
\begin{aligned}
\mathrm{d}\widehat{U}_t^0 &= \mathbb{E}[f(\widehat{\mathcal{X}}_t,\hat\theta_t)]\mathrm{d}t + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0\mathrm{d}t)\\
\mathrm{d}\widehat{\boldsymbol{U}}_t &= P^\perp_{\widehat{\boldsymbol{U}}_t}\mathbb{E}[f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\widehat{\boldsymbol{Y}}_t]\widehat{M}_{\widehat{\boldsymbol{Y}}_t}^{-1}\mathrm{d}t\\
\mathrm{d}\widehat{\boldsymbol{Y}}_t^\top &= \widehat{\boldsymbol{U}}_t^\top f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\mathrm{d}t - \frac{1}{2}\widehat{\boldsymbol{U}}_t^\top \widehat{P}_t H^\top\Gamma^{-1}H\widehat{\mathcal{X}}_t^\star\mathrm{d}t
\end{aligned}
$$

DLR-VENKF

$$
\begin{aligned}
\mathrm{d}\widehat{U}_t^0 &= \mathbb{E}_N[f(\widehat{\mathcal{X}}_t,\hat\theta_t)]\mathrm{d}t + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{U}_t^0\mathrm{d}t\\
&\qquad - \Gamma^{1/2}\mathrm{d}\mathbb{E}_N[\widehat{\boldsymbol{\mathcal{V}}}_t])\\
\mathrm{d}\widehat{\boldsymbol{U}}_t &= P^\perp_{\widehat{\boldsymbol{U}}_t}\mathbb{E}_N[f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\widehat{\boldsymbol{Y}}_t]\widehat{M}_{\widehat{\boldsymbol{Y}}_t}^{-1}\mathrm{d}t\\
\mathrm{d}\widehat{\boldsymbol{Y}}_t^\top &= \widehat{\boldsymbol{U}}_t^\top f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\mathrm{d}t - \widehat{\boldsymbol{U}}_t^\top \widehat{P}_t H^\top\Gamma^{-1}(H\widehat{\mathcal{X}}_t^\star + \Gamma^{1/2}\mathrm{d}\widehat{\mathcal{V}}_t^\star)
\end{aligned}
$$

DLR-DENKF

$$
\begin{aligned}
\mathrm{d}\widehat{U}_t^0 &= \mathbb{E}_N[f(\widehat{\mathcal{X}}_t,\hat\theta_t)]\mathrm{d}t + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0\mathrm{d}t)\\
\mathrm{d}\widehat{\boldsymbol{U}}_t &= P^\perp_{\widehat{\boldsymbol{U}}_t}\mathbb{E}_N[f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\widehat{\boldsymbol{Y}}_t]\widehat{M}_{\widehat{\boldsymbol{Y}}_t}^{-1}\mathrm{d}t\\
\mathrm{d}\widehat{\boldsymbol{Y}}_t^\top &= \widehat{\boldsymbol{U}}_t^\top f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\mathrm{d}t - \frac{1}{2}\widehat{\boldsymbol{U}}_t^\top \widehat{P}_t H^\top\Gamma^{-1}H\widehat{\mathcal{X}}_t^\star\mathrm{d}t
\end{aligned}
$$

We consider 3 ENKF variants: Vanilla (VENKF), Deterministic (DENKF), "Sequential" (SENKF)

SENKF = VENKF without $\Gamma^{1/2}\mathrm{d}\widehat{\mathcal{V}}_t$ term

Time-continuous DLR-ENKF equations

Pipeline: nonlinear equations $\rightsquigarrow$ apply DLR-SDE equations $\rightsquigarrow$ used to set up Interacting Particle System

DLR Vanilla KBP

$$
\begin{aligned}
\mathrm{d}U_t^0 &= \mathbb{E}[f(\boldsymbol{\mathcal{X}}_t,\theta_t)\,|\,\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t}]\mathrm{d}t + P_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0\mathrm{d}t)\\
\mathrm{d}\boldsymbol{U}_t &= P^{\perp}_{\boldsymbol{U}_t}\mathbb{E}[f_t^\star(\boldsymbol{\mathcal{X}}_t,\theta_t)(\boldsymbol{Y}_t M_{\boldsymbol{Y}_t}^{-1})\,|\,\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{Z}}_t}]\mathrm{d}t\\
\mathrm{d}\boldsymbol{Y}_t^\top &= \boldsymbol{U}_t^\top f_t^\star(\boldsymbol{\mathcal{X}}_t,\theta_t)\mathrm{d}t - \boldsymbol{U}_t^\top P_t H^\top \Gamma^{-1}(H\boldsymbol{\mathcal{X}}_t^\star + \Gamma^{1/2}\mathrm{d}\boldsymbol{\mathcal{V}}_t)
\end{aligned}
$$

DLR Deterministic KBP

$$
\begin{aligned}
\mathrm{d}\widehat{U}_t^0 &= \mathbb{E}[f(\widehat{\mathcal{X}}_t,\hat\theta_t)]\mathrm{d}t + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0\mathrm{d}t)\\
\mathrm{d}\widehat{\boldsymbol{U}}_t &= P^{\perp}_{\widehat{\boldsymbol{U}}_t}\mathbb{E}[f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\widehat{\boldsymbol{Y}}_t]\widehat{M}_{\widehat{\boldsymbol{Y}}_t}^{-1}\mathrm{d}t\\
\mathrm{d}\widehat{\boldsymbol{Y}}_t^\top &= \widehat{\boldsymbol{U}}_t^\top f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\mathrm{d}t - \frac{1}{2}\widehat{\boldsymbol{U}}_t^\top \widehat{P}_t H^\top \Gamma^{-1} H\widehat{\mathcal{X}}_t^\star\mathrm{d}t
\end{aligned}
$$

DLR-VENKF

$$
\begin{aligned}
\mathrm{d}\widehat{U}_t^0 &= \mathbb{E}_N[f(\widehat{\mathcal{X}}_t,\hat\theta_t)]\mathrm{d}t + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - H\widehat{U}_t^0\mathrm{d}t\\
&\qquad - \Gamma^{1/2}\mathrm{d}\mathbb{E}_N[\widehat{\boldsymbol{\mathcal{V}}}_t])\\
\mathrm{d}\widehat{\boldsymbol{U}}_t &= P^{\perp}_{\widehat{\boldsymbol{U}}_t}\mathbb{E}_N[f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\widehat{\boldsymbol{Y}}_t]\widehat{M}_{\widehat{\boldsymbol{Y}}_t}^{-1}\mathrm{d}t\\
\mathrm{d}\widehat{\boldsymbol{Y}}_t^\top &= \widehat{\boldsymbol{U}}_t^\top f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\mathrm{d}t - \widehat{\boldsymbol{U}}_t^\top \widehat{P}_t H^\top \Gamma^{-1}(H\widehat{\mathcal{X}}_t^\star + \Gamma^{1/2}\mathrm{d}\widehat{\mathcal{V}}_t^\star)
\end{aligned}
$$

DLR-DENKF

$$
\begin{aligned}
\mathrm{d}\widehat{U}_t^0 &= \mathbb{E}_N[f(\widehat{\mathcal{X}}_t,\hat\theta_t)]\mathrm{d}t + \widehat{P}_t H^\top \Gamma^{-1}(\mathrm{d}\boldsymbol{\mathcal{Z}}_t - HU_t^0\mathrm{d}t)\\
\mathrm{d}\widehat{\boldsymbol{U}}_t &= P^{\perp}_{\widehat{\boldsymbol{U}}_t}\mathbb{E}_N[f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\widehat{\boldsymbol{Y}}_t]\widehat{M}_{\widehat{\boldsymbol{Y}}_t}^{-1}\mathrm{d}t\\
\mathrm{d}\widehat{\boldsymbol{Y}}_t^\top &= \widehat{\boldsymbol{U}}_t^\top f_t^\star(\widehat{\mathcal{X}}_t,\hat\theta_t)\mathrm{d}t - \frac{1}{2}\widehat{\boldsymbol{U}}_t^\top \widehat{P}_t H^\top \Gamma^{-1} H\widehat{\mathcal{X}}_t^\star\mathrm{d}t
\end{aligned}
$$

We consider 3 ENKF variants: Vanilla (VENKF), Deterministic (DENKF), "Sequential" (SENKF)

SENKF = VENKF without $\Gamma^{1/2}\mathrm{d}\widehat{\mathcal{V}}_t$ term

Time-integration schemes

Time-integration schemes require care as additional stiffness sources

Given $\widehat{\mathcal{X}}_{t_0} = \widehat{U}^0_{t_0} + \sum_{i=1}^{R} \widehat{U}^i_{t_0} \widehat{Y}^i_{t_0}$, and denote $\widehat{\boldsymbol{Z}} = [\boldsymbol{1}, \widehat{\boldsymbol{Y}}_{t_0}]$ and $\widehat{\boldsymbol{V}} = [U^0_{t_0}, \boldsymbol{U}_{t_0}]$

Forecast step

1. Basis Update

$$\tilde{\boldsymbol{U}}(t_1) = \tilde{\boldsymbol{U}}(t_0) + \triangle t \mathbb{E}_N[f(\widehat{\boldsymbol{V}}\widehat{\boldsymbol{Z}}^\top, \hat{\theta}_{t_0})\widehat{\boldsymbol{Z}}]$$

$$\tilde{\boldsymbol{Y}}(t_1)^\top = \tilde{\boldsymbol{Y}}(t_0)^\top + \triangle t \widehat{\boldsymbol{V}}^\top f(\widehat{\boldsymbol{V}}\widehat{\boldsymbol{Z}}^\top, \hat{\theta}_{t_0})$$

$$\overline{\boldsymbol{U}} = \mathrm{ortho}([\widehat{\boldsymbol{V}}, \tilde{\boldsymbol{U}}(t_1)]) \quad \overline{\boldsymbol{Y}} = \mathrm{ortho}([\widehat{\boldsymbol{Z}}, \tilde{\boldsymbol{Y}}(t_1)])$$

2. Galerkin step

$$\boldsymbol{S}(t_1) = \boldsymbol{S}(t_0) + \triangle t \overline{\boldsymbol{U}}^\top \mathbb{E}_N[f(\widehat{\mathcal{X}}_{t_0}, \hat{\theta}_t)\overline{\boldsymbol{Y}}]$$

$$\widehat{\mathcal{X}}^{f,\mathrm{aug}}_{t_1} = \overline{\boldsymbol{U}}\boldsymbol{S}(t_1)\overline{\boldsymbol{Y}}^\top = (\widehat{U}^0_{t_1})^f + \overline{\boldsymbol{U}}\tilde{S}(t_1)\overline{\boldsymbol{Y}}^\top$$

with $\mathbb{E}_N[\overline{\boldsymbol{U}}\tilde{S}(t_1)\overline{\boldsymbol{Y}}^\top] = 0$

3. Truncation $\boldsymbol{\mathcal{T}}_R\left(\overline{\boldsymbol{U}}\tilde{S}(t_1)\overline{\boldsymbol{Y}}^\top\right) = \widehat{\boldsymbol{U}}^f_{t_1}(\widehat{\boldsymbol{Y}}_{t_1})^\top$

4. Forecasted solution $\widehat{\mathcal{X}}^f_{t_1} = (U^0_{t_1})^f + \widehat{\boldsymbol{U}}_{t_1}(\widehat{\boldsymbol{Y}}^f_{t_1})^\top$

Analysis

1. Reduced Kalman gain

$$H_{\boldsymbol{U}} = H\boldsymbol{U}_{t_1} \quad \hat{P}_{\widehat{\boldsymbol{Y}}^f_{t_1}} = \frac{1}{N-1}(\widehat{\boldsymbol{Y}}^f_{t_1})^\top \widehat{\boldsymbol{Y}}^f_{t_1}$$

$$K^{\boldsymbol{U}}_{t_1} = \hat{P}_{\widehat{\boldsymbol{Y}}^f_{t_1}} H_{\boldsymbol{U}}^\top (\Gamma + \triangle t H_{\boldsymbol{U}} \hat{P}_{\widehat{\boldsymbol{Y}}^f_{t_1}} H_{\boldsymbol{U}}^\top)^{-1}$$

2. Analysis step

$$\widehat{\mathcal{X}}^a_{t_1} = \widehat{\mathcal{X}}^f_{t_1} + \widehat{\boldsymbol{U}}_{t_1} K^{\boldsymbol{U}}_{t_1}(\Delta \boldsymbol{\mathcal{Z}}_{t_1} - H\widehat{\mathcal{X}}^f_{t_1}\triangle t - \Gamma^{1/2}\Delta\widehat{\mathcal{V}}_{t_1})$$

- stiffness from Data Assimilation ⇝ forecast/analysis scheme
- stiffness from DLR equations ⇝ BUG in forecast [CKL22]
- low-rank structure is preserved
  - analysis only performed on subspace

Rank-adaptivity easily built-in by truncating to given tolerance ϑ

Time-integration schemes

Time-integration schemes require care as additional stiffness sources

Given $\widehat{\mathcal{X}}_{t_0} = \widehat{U}^0_{t_0} + \sum_{i=1}^R \widehat{U}^i_{t_0} \widehat{Y}^i_{t_0}$, and denote $\widehat{\boldsymbol{Z}} = [\mathbf{1}, \widehat{\boldsymbol{Y}}_{t_0}]$ and $\widehat{\boldsymbol{V}} = [U^0_{t_0}, \boldsymbol{U}_{t_0}]$

Forecast step

1. Basis Update

$$\tilde{\boldsymbol{U}}(t_1) = \tilde{\boldsymbol{U}}(t_0) + \triangle t \mathbb{E}_N[f(\widehat{\boldsymbol{V}}\widehat{\boldsymbol{Z}}^\top, \hat{\theta}_{t_0})\widehat{\boldsymbol{Z}}]$$

$$\tilde{\boldsymbol{Y}}(t_1)^\top = \tilde{\boldsymbol{Y}}(t_0)^\top + \triangle t \widehat{\boldsymbol{V}}^\top f(\widehat{\boldsymbol{V}}\widehat{\boldsymbol{Z}}^\top, \hat{\theta}_{t_0})$$

$$\overline{\boldsymbol{U}} = \mathrm{ortho}([\widehat{\boldsymbol{V}}, \tilde{\boldsymbol{U}}(t_1)]) \quad \overline{\boldsymbol{Y}} = \mathrm{ortho}([\widehat{\boldsymbol{Z}}, \tilde{\boldsymbol{Y}}(t_1)])$$

2. Galerkin step

$$\boldsymbol{S}(t_1) = \boldsymbol{S}(t_0) + \triangle t \overline{\boldsymbol{U}}^\top \mathbb{E}_N[f(\widehat{\mathcal{X}}_{t_0}, \hat{\theta}_t)\overline{\boldsymbol{Y}}]$$

$$\widehat{\mathcal{X}}^{f,\mathrm{aug}}_{t_1} = \overline{\boldsymbol{U}}\boldsymbol{S}(t_1)\overline{\boldsymbol{Y}}^\top = (\widehat{U}^0_{t_1})^f + \overline{\boldsymbol{U}}\tilde{S}(t_1)\overline{\boldsymbol{Y}}^\top$$

with $\mathbb{E}_N[\overline{\boldsymbol{U}}\tilde{S}(t_1)\overline{\boldsymbol{Y}}^\top] = 0$

3. Truncation $\boldsymbol{\mathcal{T}}_R\left(\overline{\boldsymbol{U}}\tilde{S}(t_1)\overline{\boldsymbol{Y}}^\top\right) = \widehat{\boldsymbol{U}}^f_{t_1}(\widehat{\boldsymbol{Y}}_{t_1})^\top$

4. Forecasted solution $\widehat{\mathcal{X}}^f_{t_1} = (U^0_{t_1})^f + \widehat{\boldsymbol{U}}_{t_1}(\widehat{\boldsymbol{Y}}^f_{t_1})^\top$

Analysis

1. Reduced Kalman gain

$$H_{\boldsymbol{U}} = H\boldsymbol{U}_{t_1} \quad \hat{P}_{\widehat{\boldsymbol{Y}}^f_{t_1}} = \frac{1}{N-1}(\widehat{\boldsymbol{Y}}^f_{t_1})^\top \widehat{\boldsymbol{Y}}^f_{t_1}$$

$$K^{\boldsymbol{U}}_{t_1} = \hat{P}_{\widehat{\boldsymbol{Y}}^f_{t_1}} H^\top_{\boldsymbol{U}}(\Gamma + \triangle t H_{\boldsymbol{U}}\hat{P}_{\widehat{\boldsymbol{Y}}^f_{t_1}} H^\top_{\boldsymbol{U}})^{-1}$$

2. Analysis step

$$\widehat{\mathcal{X}}^a_{t_1} = \widehat{\mathcal{X}}^f_{t_1} + \widehat{\boldsymbol{U}}_{t_1} K^{\boldsymbol{U}}_{t_1}(\Delta \boldsymbol{\mathcal{Z}}_{t_1} - H\widehat{\mathcal{X}}^f_{t_1}\triangle t - \Gamma^{1/2}\Delta\widehat{\mathcal{V}}_{t_1})$$

- stiffness from Data Assimilation $\rightsquigarrow$ forecast/analysis scheme
- stiffness from DLR equations $\rightsquigarrow$ BUG in forecast [CKL22]
- low-rank structure is preserved
  - analysis only performed on subspace

Rank-adaptivity easily built-in by truncating to given tolerance ϑ

Hyperreduction

Evaluation cost of nonlinear $f_t^\star(\widehat{\mathcal{X}}_t, \hat{\theta}_t) \propto \mathcal{O}(d \cdot N) \rightsquigarrow$ no gains unless hyperreduction

- DEIM-based CUR decomposition
- approximate by evaluating at rows $\{\sigma_t^1, \dots, \sigma_t^n\}$ and columns $\{\varsigma_t^1, \dots, \varsigma_t^m\}$ with $n \ll d$ and $m \ll N$ s.t.

$$f_t^\star(\widehat{\mathcal{X}}_t, \hat{\theta}_t) \approx \left(f_t^\star(\widehat{\mathcal{X}}_t, \hat{\theta}_t)\mathbf{P}_\varsigma^\top\right) \left(\mathbf{P}_\sigma f_t^\star(\widehat{\mathcal{X}}_t, \hat{\theta}_t)\mathbf{P}_\varsigma^\top\right)^+ \left(\mathbf{P}_\sigma f_t^\star(\widehat{\mathcal{X}}_t, \hat{\theta}_t)\right)^\top$$

where $(\cdot)^+$ stands for the Moore–Penrose pseudoinverse

Algorithm ("sweeping structure")

1. select R columns $\tilde{\varsigma} = \{\varsigma_t^1, \dots, \varsigma_t^R\} = \mathrm{DEIM}(\mathrm{ortho}(\widehat{\boldsymbol{Y}}_t))$
2. select R rows $\sigma = \{\sigma_t^1, \dots, \sigma_t^R\} = \mathrm{DEIM}(\mathrm{ortho}(f_t^\star(\widehat{\mathcal{X}}_t, \hat{\theta}_t)\mathbf{P}_{\tilde{\varsigma}}^\top))$ ($\mathbf{P}_{\tilde{\varsigma}}^\top \in \mathrm{Mat}(N \times R)$ extracts $\tilde{\varsigma}$-columns)
3. select R columns $\{\varsigma_t^{R+1}, \dots, \varsigma_t^{2R}\} = \mathrm{DEIM}(\mathrm{ortho}(\mathbf{P}_\sigma f_t^\star(\widehat{\mathcal{X}}_t, \hat{\theta}_t)))$
4. set $\varsigma = \{\varsigma_t^1, \dots, \varsigma_t^{2R}\}$

Differences to [DPN+23] : σ indices determined from $f_t^\star(\widehat{\mathcal{X}}_t, \hat{\theta}_t)\mathbf{P}_\varsigma^\top$, oversampling of R columns $\{\varsigma_t^{R+1}, \dots, \varsigma_t^{2R}\}$

Fisher-KPP: experiment details

- Model
  - Domain

$$D=\{(x,y)\in\mathbb{R}^2 \mid 1\leq\|(x,y)\|_2\leq 1.5,\ x_1\geq 0, x_2\geq 0\}$$

  - Final time $T = 0.154$
  - Fisher-Kolmogorov-Petrovsky-Piskunov system

$$\partial_t u(t,x,\theta) = \mathrm{div}(\nu(x,\theta)\nabla u) + 75u(1-u)$$

  - Homogenous Neumann boundary conditions
  - Initial condition

$$u(0,x,\theta) = \exp(-(x_1-1.5)^2 - 50x_2^2)$$

- Discretisation
  - Quasi-uniform mesh $\boldsymbol{\mathcal{T}}_h \subset D$
  - Continuous FEM $\mathbb{P}_1^C(\boldsymbol{\mathcal{T}}_h)$
  - Dimension $\dim(\mathbb{P}_1^C(\boldsymbol{\mathcal{T}}_h)) = 540$
  - Time-step $\triangle t = 4.4\cdot 10^{-5}$ (3500 steps)
  - Explicit Euler

- Parameter identification problem
  - Diffusion coeff. $\nu(x,\theta) = \sqrt{2} + \sum_{i=1}^{n_\theta}\theta_i\sqrt{\lambda_i}\xi_i(x)$ for prescribed functions ξ_i
  - True parameter $\theta^{\text{true}} \in \mathbb{R}^{n_\theta}$ fixed
  - Prior $(\boldsymbol{\mathcal{X}}_0, \theta_0) \sim \delta_{u_0} \times \boldsymbol{\mathcal{N}}(\theta^{\text{true}} + p, s\boldsymbol{I})$
  - Observations

$$h^{\text{full}} = I \quad h^{\text{part}} u(x) \in \mathbb{R}^8$$

$$(h^{\text{part}}u(x))_i = \alpha\int_D \exp(-\beta\|(x-x_i, y-y_i)\|^2)u(x,y)\mathrm{d}(x,y)$$

  - Observation noise $\Gamma = \gamma\boldsymbol{I}_k$, $\gamma = 10^{-8}$
  - Ensemble size $N = 200$

Aim : identify θ^{true}

Fisher-KPP : errors and timings

	DLR			ENKF
	$R=2$	$R=5$	$R=7$	
SENKF	0.585	0.013	0.014	0.012
VENKF	0.660	0.018	0.011	0.008
DENKF	0.275	0.014	0.008	0.006

(a) Full observations

	DLR			ENKF
	$R=2$	$R=5$	$R=7$	
SENKF	0.191	0.096	0.092	0.077
VENKF	0.240	0.112	0.109	0.088
DENKF	0.232	0.113	0.111	0.086

(b) Partial observations

Table: Relative errors $\|\mathbb{E}_N[\hat{\theta}_T] - \theta^{\text{true}}\| / \|\theta^{\text{true}}\|$ (averaged over 10 runs)

	DLR			ENKF
	$R=2$	$R=5$	$R=7$	
SENKF	4.7	9.2	14.4	18.2
VENKF	4.7	9.3	13.8	18.6
DENKF	4.4	8.6	13.9	17.6

(a) Full observations

	DLR			ENKF
	$R=2$	$R=5$	$R=7$	
SENKF	5.7	11.2	15.0	19.3
VENKF	4.3	9.1	14.4	18.4
DENKF	4.2	9.0	13.9	18.0

(b) Partial observations

Table: Timings [mins]

Error sources of DLR-ENKF

Quid pro quo: what do we lose?

1. Observations assimilated only if in the range of $\widehat{\boldsymbol{U}}_t$
   - If $H = \boldsymbol{I}_d = \Gamma$, it holds
   $$K_{t_1}^{\boldsymbol{U}} \Delta \boldsymbol{\mathcal{Z}}_{t_1} = K_{t_1}^{\boldsymbol{U}} \boldsymbol{\mathcal{P}}_{\widehat{\boldsymbol{U}}_{t_1}} \Delta \boldsymbol{\mathcal{Z}}_{t_1}$$
   - technically same is true for ENKF
2. Covariance under-approximation
   - assuming that $\hat{P}_t^{\mathrm{DLR}} \approx \boldsymbol{\mathcal{T}}_R(\hat{P}_t^{\mathsf{ENKF}})$

$$\|\hat{P}_t^{\mathrm{DLR}}\|_F^2 \approx \|\boldsymbol{\mathcal{T}}_R(\hat{P}_t^{\mathsf{ENKF}})\|_F^2 = \sum_{i=1}^{R} \lambda_i^2(\hat{P}_t^{\mathsf{ENKF}}) < \sum_{i=1}^{R'} \lambda_i^2(\hat{P}_t^{\mathsf{ENKF}}) \approx \|\hat{P}_t^{\mathsf{ENKF}}\|_F^2$$

Covariance under-approximation was much more commonly observed in numerical experiments

Rank-adaptive strategies

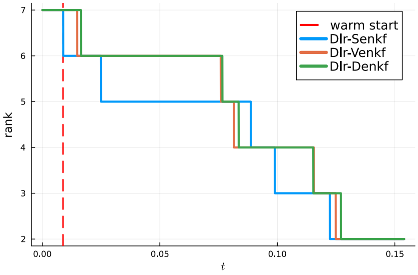


Figure: Evolution of rank r over time.

	SENKF	VENKF	DENKF
adaptive	0.030	0.011	0.007
fixed	0.007	0.012	0.005

(a) $\|\mathbb{E}_P[\hat{\theta}_T] - \theta^{\text{true}}\| / \|\theta^{\text{true}}\|$

	SENKF	VENKF	DENKF
adaptive	16.3	15.4	14.1
fixed	20.2	18.6	17.8

(b) Timings [min]

Table: Fixed/adaptive ranks (single run, full observations, truncation threshold $\vartheta = 2 \cdot 10^{-8}$)

Numerical experiment : reduced human arterial system

- Model
  - Domain : 55 arteries (1D)
  - Final time $T = 4$
  - Navier-Stokes + simplifying assumptions

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U})$$

  where
    - $\mathbf{U} = [A, u]^\top$ (cross-sectional area, velocity)
    - $$\mathbf{F}(\mathbf{U}) = \begin{bmatrix} Au \\ \frac{u^2}{2} + \frac{p_{\text{ext}} + \beta(\sqrt{A} - \sqrt{A_0})}{\rho} \end{bmatrix}$$

    where β elastic coefficient
- State+parameter identification problem
  - True parameter $\theta^{\text{true}} = [\beta_1, \beta_{13}]$ (elastic coefficients of first two aortas)
  - Observations

$$H\mathbf{U} = [\mathbf{U}_5^{\text{middle}}, \mathbf{U}_9^{\text{middle}}, \mathbf{U}_{46}^{\text{middle}}]$$

Aim: identify $\boldsymbol{\mathcal{X}}_t^{\text{true}}$ and θ^{true}

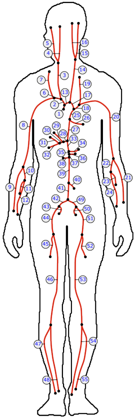


Figure: Arterial tree consisting of 55 main human arteries

Reduced human arterial system : experiment details

- Model
  - Domain : 55 arteries (1D)
  - Final time $T = 4$
  - Navier-Stokes + simplifying assumptions

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{S}(\mathbf{U})$$

  where
    - $\mathbf{U} = [A, u]^\top$ (cross-sectional area, velocity)
    - $$\mathbf{F}(\mathbf{U}) = \begin{bmatrix} Au \\ \frac{u^2}{2} + \frac{p_{\text{ext}} + \beta(\sqrt{A} - \sqrt{A_0})}{\rho} \end{bmatrix}$$

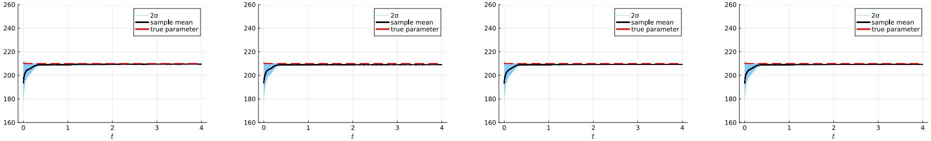
    where β elastic coefficient
  - Hyperbolic system
  - Initial condition : periodic inflow
  - Nonlinear system at bifurcation nodes
  - Non-reflecting BC at terminal conditions
- Discretisation
  - Mesh size $\Delta x = 2 \cdot 10^{-3}$
  - Second order FV type (3542 DOFs)
    - MUSCL reconstruction
    - local Lax-Friedrich numerical flux
  - Time-step $\triangle t = 5 \cdot 10^{-5}$ $(80000$ steps)
  - Explicit Euler
- State+parameter identification problem
  - True parameter $\theta^{\text{true}} = [\beta_1, \beta_{13}]$ (elastic coefficients of first two aortas)
  - Prior $(\boldsymbol{\mathcal{X}}_0, \theta_0) \sim \delta_{u_0} \times \boldsymbol{\mathcal{N}}(\theta^{\text{true}} + p, s\boldsymbol{I})$
  - Observations

$$H\mathbf{U} = [\mathbf{U}_5^{\text{middle}}, \mathbf{U}_9^{\text{middle}}, \mathbf{U}_{46}^{\text{middle}}]$$

  - Observation noise $\Gamma = \begin{bmatrix} 10^{-2}\boldsymbol{I}_3 & \\ & 10^{-7}\boldsymbol{I}_3 \end{bmatrix}$
  - Ensemble size $P = 100$

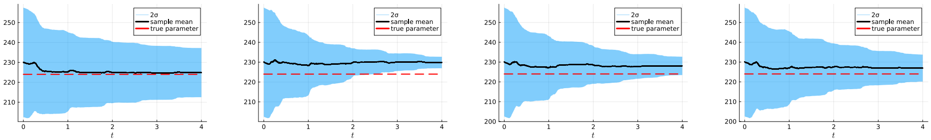
Aim : identify $\boldsymbol{\mathcal{X}}_t^{\text{true}}$ and θ^{true}

Reduced human arterial system: identifying β_1 and β_{13}



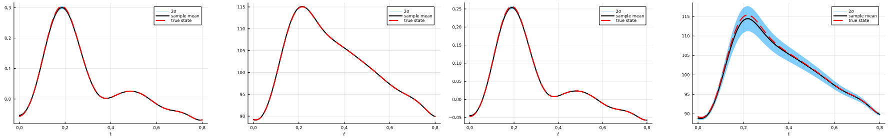
(a) ENKF (b) DLR $R = 10$ (c) DLR $R = 15$ (d) DLR $R = 20$

Figure: (DLR-)DENKF for β_1 with $P = 100$ particles



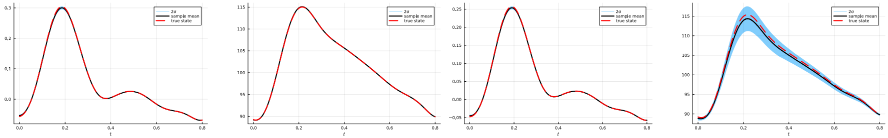
(a) ENKF (b) DLR $R = 10$ (c) DLR $R = 15$ (d) DLR $R = 20$

Figure: (DLR-)DENKF for β_{13} with $P = 100$ particles



(a) Q_1 ($\mathrm{L\,s^{-1}}$) (b) p_1 (mmHg) (c) Q_{13} ($\mathrm{L\,s^{-1}}$) (d) p_{13} (mmHg)

Figure: Full Order Model DENKF state estimation with $P = 100$ particles



(a) Q_1 ($\mathrm{L\,s^{-1}}$) (b) p_1 (mmHg) (c) Q_{13} ($\mathrm{L\,s^{-1}}$) (d) p_{13} (mmHg)

Figure: DLR-DENKF ($R = 10$) state estimation with $P = 100$ particles